AoPS

WOOT

Art of Problem Solving

Worldwide Online Olympiad Training

2017-2018

Syllabus

1. Special Functions

This class will cover problems involving algebraic functions other than polynomials, such as square roots, the floor function, and logarithms.

Example Problem: Let n be a positive integer. Show that for any real number x,

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor.$$

Example Problem: Show that for every positive integer n,

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor.$$

2. Trigonometry/Complex Numbers A
3. Trigonometry/Complex Numbers B

These classes will explore the relationship between trigonometric identities and complex numbers.
Example Problem: Find the positive integer n such that

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{n} = \frac{\pi}{4}.$$

(2008 AIME I)

Example Problem: Show that

$$\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}.$$

4. Algebraic Manipulations

This class will cover problems where skillful manipulation of algebra is required, involving techniques like substitutions, factorizations, and applications of common identities.

Example Problem: The variables a, b, c, d, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes? (IMO, 1974)

Example Problem: Let u and v be real numbers such that

$$(u + u^2 + u^3 + \cdots + u^8) + 10u^9 = (v + v^2 + v^3 + \cdots + v^{10}) + 10v^{11} = 8.$$

Determine, with proof, which of the two numbers, u or v, is larger. (USAMO, 1989)

5. Geometry of the Circle A
6. Geometry of the Circle B

These classes will cover results regarding the geometry of the circle, including cyclic quadrilaterals and cyclic polygons, Ptolemy's theorem, power of a point, the radical axis, and the radical center.

Example Problem: Let C be a circle and P a given point in the plane. Each line through P which intersects C determines a chord of C. Show that the midpoints of these chords lie on a circle. (Canada, 1991)

Example Problem: Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD. (IMO, 1999)

7. Geometric Transformations

This class will introduce the use and applications of geometric transformations, such as homothety and spiral similarity.

Example Problem: Circles Γ_1 and Γ_2 are internally tangent at P, with Γ_1 as the larger circle. A line intersects Γ_1 at A and D, and Γ_2 at B and C, so that points A, B, C, and D lie on the line in that order. Show that $\angle APB = \angle CPD$.

Example Problem: On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45°$, $\angle BCP = \angle ACQ = 30°$, $\angle ABR = \angle BAR = 15°$. Prove that $\angle QRP = 90°$ and $QR = RP$. (IMO, 1975)

8. Locus/Construction

This class will examine locus and construction problems in geometry.

Example Problem: Let ABC be an equilateral triangle. For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle. (Ireland, 1997)

Example Problem: Two circles Q and R in the plane intersect at A and Z. From A, a point S goes around Q while a point T traverses R. Both points travel in the counterclockwise direction, proceed at constant speeds (not necessarily equal to each other), start together and finish together. Prove the remarkable fact that there exists a fixed point P in the plane with the property that, at every instant of the motions, it is the same distance from S as it is from T. (IMO, 1979)

9. Invariants & Monovariants

This class will introduce the topics of invariants and monovariants.

Example Problem: Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any n the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant. (USAMO, 2007)

Example Problem: To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps. (IMO, 1986)

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10. Combinatorial Games

This class will show how to analyze combinatorial games.

Example Problem: A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking **either**

(a) one bean from a heap, provided at least two beans are left behind in that heap, **or**

(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy. (Putnam, 1995)

Example Problem: The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy. (USAMO, 1999)

11. Algorithms

This class will look at problems involving algorithms and other kinds of processes.

Example Problem: Let p_1, p_2, p_3, ... be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence x_0, x_1, x_2, ... eventually becomes 0. (USAMO, 1997)

Example Problem: On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board. (IMO, 1993)

12. Combinatorial Geometry

This class will cover problems in combinatorial geometry and discuss techniques like discrete continuity and the convex hull that are often used to solve these problems.

Example Problem: In a plane a set of $n \geq 3$ points is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n. (IMO, 1965)

Example Problem: Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines. (USAMO, 2005)

13. Digits and Bases

This class will cover problems and techniques dealing with digit representations of numbers, possibly in bases other than 10. It will also apply these techniques to problems that can be modeled by base representations.

Example Problem: Show that the equation

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor + \lfloor 16x \rfloor + \lfloor 32x \rfloor = 12345$$

has no real solutions. (Canada, 1981)

Example Problem: A function f is defined on the positive integers by

$$\begin{aligned} f(1) = 1, \quad f(3) &= 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{aligned}$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988, for which $f(n) = n$. (IMO, 1988)

14. Primes

This class will look at properties of primes, particularly powers of primes.

Example Problem: Let p be a prime number. Show that

$$\binom{2p}{p} \equiv 2 \pmod{p}.$$

Example Problem: Find the highest positive integer k for which 1991^k divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

(IMO Short List, 1991)

15. Linear Number Theory

This class will cover advanced topics relating to the Euclidean Algorithm, the Chinese Remainder Theorem, the Frobenius Coin Problem (also known as the Chicken McNugget Theorem), and Farey Fractions.

Example Problem: Start with the empty set S, and repeatedly do the following: choose a number k that cannot be written as the sum of elements of S (elements may be repeated in the sum any number of times), and add k to S. Show that we will eventually be unable to add any more elements to S while S is still finite.

Example Problem: Prove that there are n consecutive positive integers, none of which are a sum of two perfect squares.

16. Vieta Jumping and LTE

This class will cover Vieta Jumping and Lifting the Exponent, two results in advanced olympiad number theory that are useful in many problems.

Example Problem: Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$. (IMO, 2007)

Example Problem: Let b, m, n be positive integers such that $b > 1$ and $m \neq n$. Prove that if $b^m - 1$ and $b^n - 1$ have the same prime divisors, then $b + 1$ is a power of 2. (IMO Short List, 1997)