National Olympiad Second Round 2014

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Day 1 November 16th

1 In a bag there are 1007 black and 1007 white balls, which are randomly numbered 1 to 2014. In every step we draw one ball and put it on the table; also if we want to, we may choose two different colored balls from the table and put them in a different bag. If we do that we earn points equal to the absolute value of their differences. How many points can we guarantee to earn after 2014 steps?

2 Find all all positive integers x,y,and z satisfying the equation $x^3 = 3^y7^z + 8$

3 Let D, E, F be points on the sides BC, CA, AB of a triangle ABC, respectively such that the lines AD, BE, CF are concurrent at the point P. Let a line ℓ through A intersect the rays $[DE$ and $[DF$ at the points Q and R, respectively. Let M and N be points on the rays $[DB$ and $[DC$, respectively such that the equation

$$\frac{QN^2}{DN} + \frac{RM^2}{DM} = \frac{(DQ + DR)^2 - 2 \cdot RQ^2 + 2 \cdot DM \cdot DN}{MN}$$

holds. Show that the lines AD and BC are perpendicular to each other.

Day 2 November 17th

4 Let P and Q be the midpoints of non-parallel chords k_1 and k_2 of a circle ω, respectively. Let the tangent lines of ω passing through the endpoints of k_1 intersect at A and the tangent lines passing through the endpoints of k_2 intersect at B. Let the symmetric point of the orthocenter of triangle ABP with respect to the line AB be R and let the feet of the perpendiculars from R to the lines AP, BP, AQ, BQ be R_1, R_2, R_3, R_4, respectively. Prove that

$$\frac{AR_1}{PR_1} \cdot \frac{PR_2}{BR_2} = \frac{AR_3}{QR_3} \cdot \frac{QR_4}{BR_4}$$

5 Find all natural numbers n for which there exist non-zero and distinct real numbers $a_1, a_2, \ldots, a_n$ satisfying

$$\left\{a_i + \frac{(-1)^i}{a_i} \,\middle|\, 1 \le i \le n\right\} = \{a_i \mid 1 \le i \le n\}.$$

6 5 airway companies operate in a country consisting of 36 cities. Between any pair of cities exactly one company operates two way flights. If some air company operates between cities A, B and B, C we say that the triple A, B, C is *properly-connected*. Determine the largest possible value of k such that no matter how these flights are arranged there are at least k properly-connected triples.
