for some positive number a. Joon graphs a function of the form

$$f(x) = \sin(bx + c)$$

for some positive number b and real number c. They produce the same graph, which is shown below.

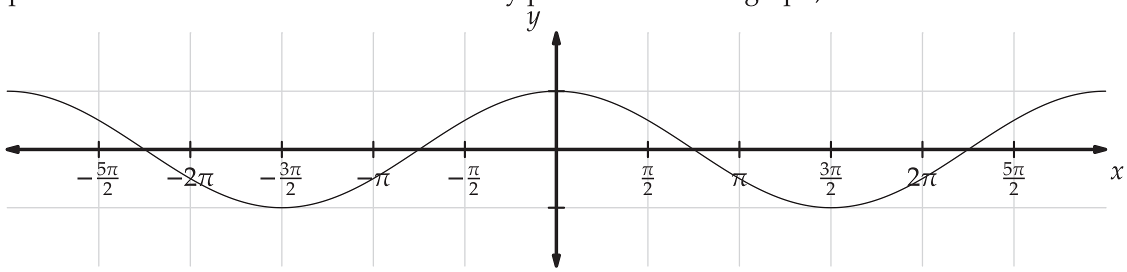

Figure 2.22: Graph for Exercise 2.4.7

(a) What is a?

(b) What is b?

(c) Find three possible values of c.

2.4.8 Is the function $f(x) = 2\sin x - \tan x$ periodic? If so, what is the period?

2.5 Inverse Trig Functions

So far in this chapter, we've mostly concerned ourselves with finding the output of trigonometric functions given certain inputs. In this section, we turn this around and try to find the inputs we must give to trig functions to produce certain desired outputs.

Problems

Problem 2.28: A cart is attached to a spring that is connected to a wall. The spring is compressed so that the cart is placed 4 feet away from the wall. The cart is then released, so that it rolls back and forth away from and towards the wall as the spring stretches and contracts. The distance from the cart to the wall varies sinusoidally with time. Suppose the distance between the cart and the wall varies between 4 feet and 8 feet, and that it takes 6 seconds for the cart to go from the point closest to the wall to the point that is farthest from the wall.

(a) Find a function that describes how far the cart is from the wall t seconds after it is released.

(b) How many seconds pass before the cart is exactly 7 feet from the wall for the first time?

(c) Find all values of t such that the cart is 7 feet from the wall after t seconds.

Problem 2.29:

(a) Explain why the function $f(x) = \cos x$ does not have an inverse.

(b) Suppose we define a function $g(x) = \cos x$ and restrict the domain of g to an interval $[a, b]$, where a and b are constants, such that g does have an inverse. What are valid choices for a and b? What are *good choices* for a and b?

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Solution for Problem 2.28:

(a) Because the distance between the cart and the wall varies sinusoidally with time, this distance is

$$f(t) = a\sin(bt + c) + d,$$

where a, b, c, and d are constants. The middle point of the cart's path is 6 feet from the wall, and it varies 2 feet from this point in both directions. So, the amplitude of $a\sin(bt + c)$ is 2, and $f(t)$ must equal 6 when $a\sin(bt + c)$ is 0 (when the cart is at the midpoint of its path). Therefore, the function is $f(t) = 2\sin(bt + c) + 6$ for some values of b and c.

The cart starts 4 feet away from the wall, so we must have $f(0) = 4$. So, we must have $\sin(bt + c) = -1$ when $t = 0$, which means $\sin c = -1$. This means we can let $c = -\frac{\pi}{2}$, and we have

$$f(t) = 2\sin\left(bt - \frac{\pi}{2}\right) + 6.$$

(Note that we could let $c = \frac{3\pi}{2}$, or $-\frac{5\pi}{2}$, or any number that is a multiple of 2π from $-\frac{\pi}{2}$.)

Finally, we turn to the last piece of information we have: the cart goes from the closest point to the wall to the farthest point in 6 seconds. This means that it goes from the closest point to the farthest and back to the closest again in 12 seconds, so its period is 12 seconds. Since the period of $f(t)$ is $\frac{2\pi}{b}$ when b is positive, we must have $b = \pi/6$, and we have

$$f(t) = 2\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 6.$$

(b) We seek the smallest time t such that we have $f(t) = 7$, so we have

$$2\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 6 = 7.$$

Isolating the sine function, we have

$$\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) = \frac{1}{2}.$$

We now need to find an angle whose sine equals $\frac{1}{2}$. Fortunately, we've seen $\sin\theta = \frac{1}{2}$ enough times by now to remember that $\sin\frac{\pi}{6} = \frac{1}{2}$. Therefore, if $\frac{\pi}{6}t - \frac{\pi}{2} = \frac{\pi}{6}$, then the equation above is satisfied. Solving for t gives $t = 4$. We still have to show that no smaller t is possible. The cart goes from 4 feet from the wall to 8 feet from the wall during the first 6 seconds, and the distance between the cart and the wall is strictly increasing during that time. This means the cart can only be 7 feet from the wall once in those first 6 seconds. Since the cart is 7 feet from the wall when $t = 4$, this is smallest possible positive value of t.

(c) What's wrong with this solution:

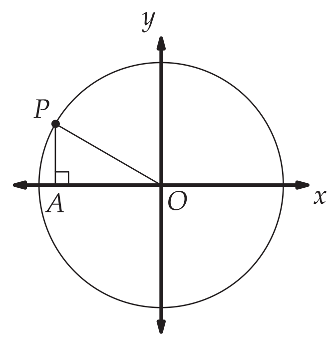
Bogus Solution: The period of sine is 2π, so the value of $\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right)$ repeats every time the argument increases by 2π. Since we have $\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) = \frac{1}{2}$ when $\frac{\pi}{6}t - \frac{\pi}{2} = \frac{\pi}{6}$, we must have $\frac{\pi}{6}t - \frac{\pi}{2} = \frac{\pi}{6} + k(2\pi)$ for some nonnegative integer k. (No going backwards in time! We must have $k \geq 0$.) Therefore, the times at which the cart is 7 feet from the wall are $t = 4 + 12k$, for any nonnegative integer k.

We see the error in this reasoning by stepping back from the math and thinking about the problem.

WARNING!! When using math to model a problem, don't get so wrapped up in the math that you overlook what the problem is about. Keeping the original problem in mind can help you catch errors in your thinking.

The cart is 7 feet from the wall twice during each period—once as it moves away from the wall and once as it approaches the wall. The Bogus Solution only includes the times that are one full period from the first time the cart is 7 feet from the wall, which are the times during which the cart is moving away from the wall. (We also could have caught this mistake by thinking about the unit circle—there are two points on the unit circle with y-coordinate equal to $\frac{1}{2}$, so there are two angles θ in each period of sine such that $\sin\theta = \frac{1}{2}$.)

Here are a couple ways to finish the problem:



Solution 1: Use Trigonometry. We can use the unit circle to find the other set of angles whose sine is $\frac{1}{2}$. In the diagram, point P is the second quadrant point on the unit circle with y-coordinate $\frac{1}{2}$. We draw altitude $\overline{PA}$ as shown; the length of this altitude equals the y-coordinate of P, so $PA = \frac{1}{2}$. Since leg $\overline{PA}$ of right triangle PAO has length $\frac{1}{2}$ the hypotenuse of PAO, the triangle is a 30-60-90 right triangle with $\angle POA = 30^\circ = \frac{\pi}{6}$. Therefore, P is the terminal point of $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

So, in addition to the values of t that we found in the Bogus Solution, we also have $\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) = \frac{1}{2}$ when $\frac{\pi}{6}t - \frac{\pi}{2} = \frac{5\pi}{6} + (2\pi)k$, for any nonnegative integer k. Solving for t gives $t = 8 + 12k$, where k is a nonnegative integer. Combining this with the earlier values we found, the cart is 7 feet from the wall when $t = 4 + 12k$ seconds or $t = 8 + 12k$ seconds, for any nonnegative integer k.

Solution 2: Use Symmetry. Once again, we help ourselves by thinking about the original problem rather than staying focused on our equations. The cart goes from 4 feet away to 7 feet away in 4 seconds. Since it takes 6 seconds to go from 4 feet away to 8 feet away, it will take $6 - 4 = 2$ more seconds for the cart to complete its first trip from nearest point to farthest point. By symmetry, it will then take 2 more seconds to return to the point 7 feet from the wall, so $4 + 2 + 2 = 8$ seconds after the cart is released, it will again be 7 feet from the wall.

Since the period of the cart is 12 seconds and the two times in the first period at which the cart is 7 feet from the wall are at $t = 4$ seconds and $t = 8$ seconds, the values of t for which the cart are 7 feet from the wall are $t = 4 + 12k$ seconds or $t = 8 + 12k$ seconds, for any nonnegative integer k.

□

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