AMC 10 2022

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– A

– November 10th, 2022

1 What is the value of
$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}?$$

(A) $\frac{31}{10}$ **(B)** $\frac{49}{15}$ **(C)** $\frac{33}{10}$ **(D)** $\frac{109}{33}$ **(E)** $\frac{15}{4}$

2 Mike cycled 15 laps in 57 minutes. Assume he cycled at a constant speed throughout. Approximately how many laps did he complete in the first 27 minutes?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

3 The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

4 In some countries, automobile fuel efficiency is measured in liters per 100 kilometers while other countries use miles per gallon. Suppose that 1 kilometer equals m miles, and 1 gallon equals ℓ liters. Which of the following gives the fuel efficiency in liters per 100 kilometers for a car that gets x miles per gallon?

(A) $\frac{x}{100\ell m}$ **(B)** $\frac{x\ell m}{100}$ **(C)** $\frac{\ell m}{100x}$ **(D)** $\frac{100}{x\ell m}$ **(E)** $\frac{100\ell m}{x}$

5 Square $ABCD$ has side length 1. Point P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?

(A) $\frac{\sqrt{2}}{3}$ **(B)** $\frac{1}{2}$ **(C)** $2-\sqrt{2}$ **(D)** $1-\frac{\sqrt{2}}{4}$ **(E)** $\frac{2}{3}$

6 Which expression is equal to $\left|a-2-\sqrt{(a-1)^2}\right|$ for $a<0$?

(A) $3-2a$ **(B)** $1-a$ **(C)** 1 **(D)** $a+1$ **(E)** 3

7 The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

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8 A data set consists of 6 (not distinct) positive integers: $1, 7, 5, 2, 5$, and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?

(A) 10 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 40

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9 A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?

(A) 120 **(B)** 270 **(C)** 360 **(D)** 540 **(E)** 720

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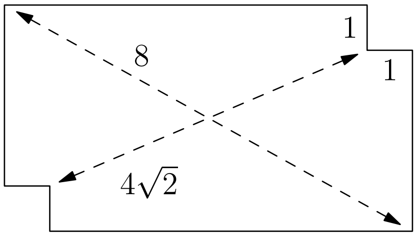
10 Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be $4\sqrt{2}$ centimeters, as shown below. What is the area of the original index card?



(A) 14 **(B)** $10\sqrt{2}$ **(C)** 16 **(D)** $12\sqrt{2}$ **(E)** 18

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11 Ted mistakenly wrote $2^m \cdot \sqrt{\frac{1}{4096}}$ as $2 \cdot \sqrt[m]{\frac{1}{4096}}$. What is the sum of all real numbers m for which these two expressions have the same value?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

12 On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

(A) 7 **(B)** 12 **(C)** 21 **(D)** 27 **(E)** 31

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13 Let $\triangle ABC$ be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC$. The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point D. Suppose $BP = 2$ and $PC = 3$. What is AD ?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

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14 What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

(A) 108 **(B)** 120 **(C)** 126 **(D)** 132 **(E)** 144

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15 Quadrilateral $ABCD$ with side lengths $AB = 7, BC = 24, CD = 20, DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where $a, b,$ and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

(A) 260 **(B)** 855 **(C)** 1235 **(D)** 1565 **(E)** 1997

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16 The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism. A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

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17 How many three-digit positive integers $\underline{a}\ \underline{b}\ \underline{c}$ are there whose nonzero digits a, b, and c satisfy

$$0.\overline{\underline{a}\ \underline{b}\ \underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$$

(The bar indicates repetition, thus $0.\overline{\underline{a}\,\underline{b}\,\underline{c}}$ in the infinite repeating decimal $0.\underline{a}\,\underline{b}\,\underline{c}\,\underline{a}\,\underline{b}\,\underline{c}\,\cdots$)

(A) 9 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 14

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18 Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations transformations $T_1, T_2, T_3, \dots, T_n$ returns the point $(1,0)$ back to itself?

(A) 359 **(B)** 360 **(C)** 719 **(D)** 720 **(E)** 721

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19 Define L_n as the least common multiple of all the integers from 1 to n inclusive. There is a unique integer h such that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\dots+\frac{1}{17}=\frac{h}{L_{17}}$. What is the remainder when h is divided by 17?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

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20 A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are 57, 60, and 91. What is the fourth term of this sequence?

(A) 190 **(B)** 194 **(C)** 198 **(D)** 202 **(E)** 206

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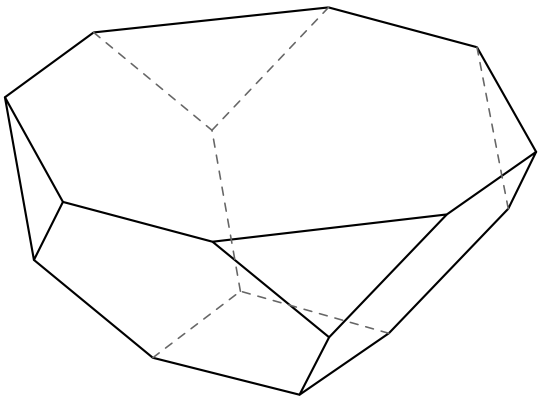
21 A bowl is formed by attaching four regular hexagons of side 1 to a square of side 1. The edges of adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the bowl?



(A) 6 **(B)** 7 **(C)** $5+2\sqrt{2}$ **(D)** 8 **(E)** 9

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22 Suppose that 13 cards numbered $1,2,3,\dots,13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

$\boxed{7}$ $\boxed{11}$ $\boxed{8}$ $\boxed{6}$ $\boxed{4}$ $\boxed{5}$ $\boxed{9}$ $\boxed{12}$ $\boxed{1}$ $\boxed{13}$ $\boxed{10}$ $\boxed{2}$ $\boxed{3}$

(A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191

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23 Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3,$ and $PD = 4.$ What is $\frac{BC}{AD}$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

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24 How many strings of length 5 formed from the digits 0,1,2,3,4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

(A) 500 **(B)** 625 **(C)** 1089 **(D)** 1199 **(E)** 1296

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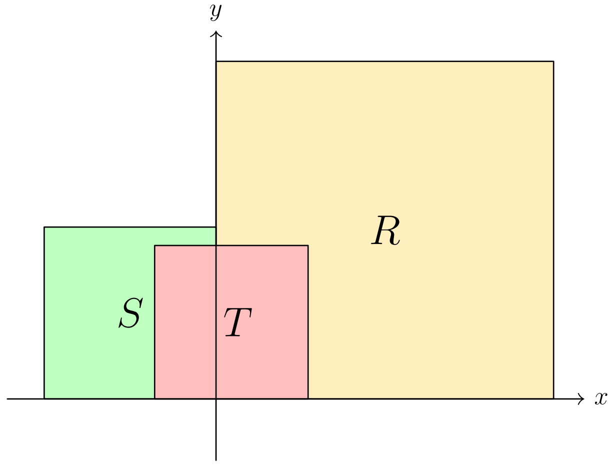
25 Let R, S, and T be squares that have vertices at lattice points (i.e., points whose coordinates are both integers) in the coordinate plane, together with their interiors. The bottom edge of each square is on the x-axis. The left edge of R and the right edge of S are on the y-axis, and R contains $\frac{9}{4}$ as many lattice points as does S. The top two vertices of T are in $R \cup S$, and T contains $\frac{1}{4}$ of the lattice points contained in $R \cup S$. See the figure (not drawn to scale).



The fraction of lattice points in S that are in $S \cap T$ is 27 times the fraction of lattice points in R that are in $R \cap T$. What is the minimum possible value of the edge length of R plus the edge length of S plus the edge length of T?

(A) 336 **(B)** 337 **(C)** 338 **(D)** 339 **(E)** 340

– B

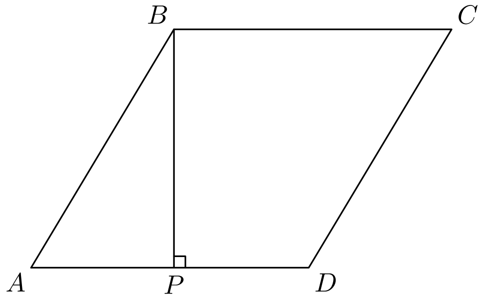
– November 16th, 2022

1 Define $x \diamond y$ to be $|x - y|$ for all real numbers x and y. What is the value of

$$(1 \diamond (2 \diamond 3)) - ((1 \diamond 2) \diamond 3)?$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

2 In rhombus $ABCD$, point P lies on segment $\overline{AD}$ such that $BP \perp AD$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$?



(A) $3\sqrt{5}$ **(B)** 10 **(C)** $6\sqrt{5}$ **(D)** 20 **(E)** 25

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3 How many three-digit positive integers have an odd number of even digits?

(A) 150 **(B)** 250 **(C)** 350 **(D)** 450 **(E)** 550

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4 A donkey suffers an attack of hiccups and the first hiccup happens at 4:00 one afternoon. Suppose that the donkey hiccups regularly every 5 seconds. At what time does the donkey's 700th hiccup occur?

(A) 15 seconds after 4:58

(B) 20 seconds after 4:58

(C) 25 seconds after 4:58

(D) 30 seconds after 4:58

(E) 35 seconds after 4:58

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5 What is the value of $\frac{(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})}{\sqrt{(1-\frac{1}{3^2})(1-\frac{1}{5^2})(1-\frac{1}{7^2})}}$?

(A) $\sqrt{3}$ **(B)** 2 **(C)** $\sqrt{15}$ **(D)** 4 **(E)** $\sqrt{105}$

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6 How many of the first ten numbers of the sequence 121, 11211, 1112111, ... are prime numbers?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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7 For how many values of the constant k will the polynomial $x^2+kx+36$ have two distinct integer roots?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 14 **(E)** 16

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8 Consider the following 100 sets of 10 elements each:

$$\begin{aligned}&\{1,2,3,\cdots,10\},\\&\{11,12,13,\cdots,20\},\\&\{21,22,23,\cdots,30\},\\&\vdots\\&\{991,992,993,\cdots,1000\}.\end{aligned}$$

How many of these sets contain exactly two multiples of 7?

(A)40 **(B)**42 **(C)**42 **(D)**49 **(E)**50

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9 The sum

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{2021}{2022!}$$

can be expressed as $a-\frac{1}{b!}$, where a and b are positive integers. What is $a+b$?

(A) 2020 **(B)** 2021 **(C)** 2022 **(D)** 2023 **(E)** 2024

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10 Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

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11 All the high schools in a large school district are involved in a fundraiser selling T-shirts. Which of
the choices below is logically equivalent to the statement ”No school bigger than Euclid HS sold more
T-shirts than Euclid HS”?

(**A**) All schools smaller than Euclid HS sold fewer T-shirts than Euclid HS. (**B**) No school that sold more T-shirts than Euclid HS is bigger than Euclid HS. (**C**) All schools bigger than Euclid HS sold fewer T-shirts than Euclid HS. (**D**) All schools that sold fewer T-shirts than Euclid HS are smaller than Euclid HS. (**E**) All schools smaller than Euclid HS sold more T-shirts than Euclid HS.

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12 A pair of fair 6-sided dice is rolled n times. What is the least value of n such that the probability that the sum of the numbers face up on a roll equals 7 at least once is greater than $\frac{1}{2}$?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

---

13 The positive difference between a pair of primes is equal to 2, and the positive difference between the cubes of the two primes is 31106. What is the sum of the digits of the least prime that is greater than those two primes?

(A) 8 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 16

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14 Suppose that S is a subset of $\{1,2,3,...,25\}$ such that the sum of any two (not necessarily distinct) elements of S is never an element of S. What is the maximum number of elements S may contain?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

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15 Let S_n be the sum of the first n term of an arithmetic sequence that has a common difference of 2. The quotient $\frac{S_{3n}}{S_n}$ does not depend on n. What is S_{20}?

(A) 340 **(B)** 360 **(C)** 380 **(D)** 400 **(E)** 420

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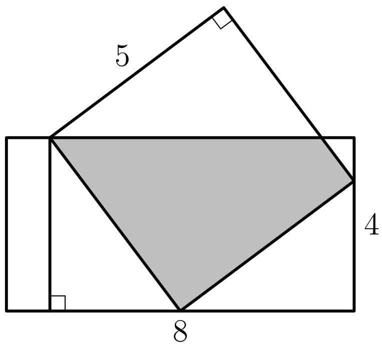
16 The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



(A) $15\frac{1}{8}$ **(B)** $15\frac{3}{8}$ **(C)** $15\frac{1}{2}$ **(D)** $15\frac{5}{8}$ **(E)** $15\frac{7}{8}$

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17 One of the following numbers is not divisible by any prime number less than 10. Which is it?
(A) $2^{606}-1$ (B) $2^{606}+1$ (C) $2^{607}-1$ (D) $2^{607}+1$ (E) $2^{607}+3^{607}$

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18 Consider systems of three linear equations with unknowns x, y, and z,

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

where each of the coefficients is either 0 or 1 and the system has a solution other than $x=y=z=0$. For example, one such system is $\{1x+1y+0z=0, 0x+1y+1z=0, 0x+0y+0z=0\}$ with a nonzero solution of $\{x,y,z\}=\{1,-1,1\}$. How many such systems are there? (The equations in a system need not be distinct, and two systems containing the same equations in a different order are considered different.)

(A) 302 **(B)** 338 **(C)** 340 **(D)** 343 **(E)** 344

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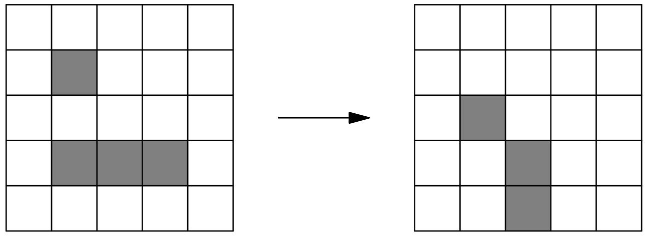
19 Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

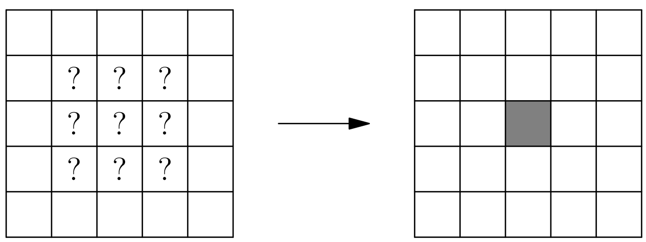
- Any filled square with two or three filled neighbors remains filled.
- Any empty square with exactly three filled neighbors becomes a filled square.
- All other squares remain empty or become empty.

A sample transformation is shown in the figure below.



Initial Transformed

Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



Initial Transformed

(A) 14 (B) 18 (C) 22 (D) 26 (E) 30

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20 Let $ABCD$ be a rhombus with $\angle ADC = 46^\circ$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

(A) 110 **(B)** 111 **(C)** 112 **(D)** 113 **(E)** 114

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21 Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

(A) 10 **(B)** 13 **(C)** 19 **(D)** 20 **(E)** 23

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22 Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations $x^2 + y^2 = 4$, $x^2 + y^2 = 64$, and $(x - 5)^2 + y^2 = 3$. What is the sum of the areas of all circles in S?

(A) 48π **(B)** 68π **(C)** 96π **(D)** 102π **(E)** 136π

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23 Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3,$ Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

(A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{5}{6}$

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24 Consider functions f that satisfy $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all real numbers x and y. Of all such functions that also satisfy the equation $f(300) = f(900)$, what is the greatest possible value of

$$f(f(800)) - f(f(400))?$$

(A) 25 **(B)** 50 **(C)** 100 **(D)** 150 **(E)** 200

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25 Let x_0, x_1, x_2, $\cdots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \ge 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

(A) 6 **(B)** 7 **(C)** 12 **(D)** 14 **(E)** 15
