AIME Problems 2006

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– I

– March 7th

1 In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

2 Let set $\mathcal{A}$ be a 90-element subset of $\{1, 2, 3, \ldots, 100\}$, and let S be the sum of the elements of $\mathcal{A}$. Find the number of possible values of S.

3 Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

4 Let N be the number of consecutive 0's at the right end of the decimal representation of the product $1! \times 2! \times 3! \times 4! \cdots 99! \times 100!$. Find the remainder when N is divided by 1000.

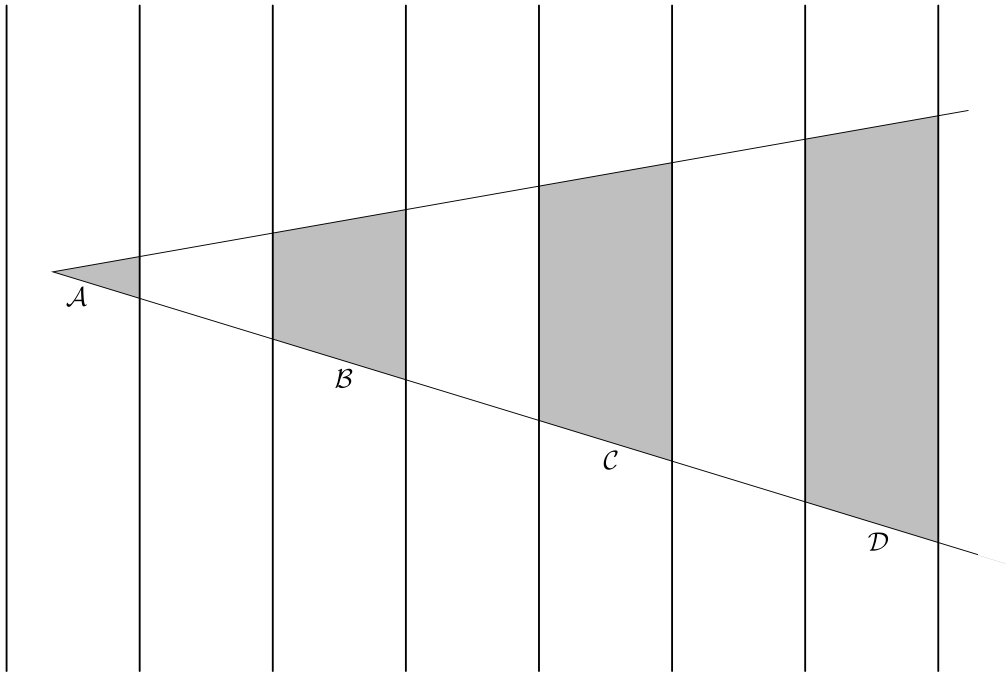
5 The number

$$\sqrt{104\sqrt{6} + 468\sqrt{10} + 144\sqrt{15} + 2006}$$

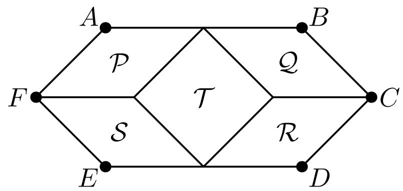
can be written as $a\sqrt{2} + b\sqrt{3} + c\sqrt{5}$, where a, b, and c are positive integers. Find $a \cdot b \cdot c$.

6 Let $\mathcal{S}$ be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$ where a, b, c are distinct digits. Find the sum of the elements of $\mathcal{S}$.

7 An angle is drawn on a set of equally spaced parallel lines as shown. The ratio of the area of shaded region $\mathcal{C}$ to the area of shaded region $\mathcal{B}$ is $11/5$. Find the ratio of shaded region $\mathcal{D}$ to the area of shaded region $\mathcal{A}$.

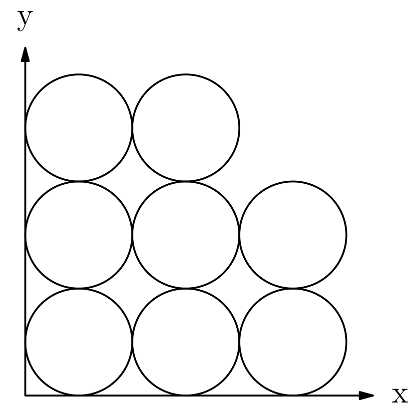


8 Hexagon $ABCDEF$ is divided into four rhombuses, $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, as shown. Rhombuses $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, and $\mathcal{S}$ are congruent, and each has area $\sqrt{2006}$. Let K be the area of rhombus $\mathcal{T}$. Given that K is a positive integer, find the number of possible values for K.



9 The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006,$ find the number of possible ordered pairs (a, r).

10 Eight circles of diameter 1 are packed in the first quadrant of the coordinte plane as shown. Let region $\mathcal{R}$ be the union of the eight circular regions. Line l, with slope 3, divides $\mathcal{R}$ into two regions of equal area. Line l's equation can be expressed in the form $ax = by + c$, where $a, b,$ and c are positive integers whose greatest common divisor is 1. Find $a^2 + b^2 + c^2$.



11 A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:
• Any cube may be the bottom cube in the tower. • The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

12 Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

13 For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer n, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer n less than 1000 such that S_n is a perfect square.

14 A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $\frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m + \sqrt{n} \rfloor$. (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

15 Given that a sequence satisfies $x_0 = 0$ and $|x_k| = |x_{k-1} + 3|$ for all integers $k \geq 1$, find the minimum possible value of $|x_1 + x_2 + \cdots + x_{2006}|$.

– II

– March 22nd

1 In convex hexagon $ABCDEF$, all six sides are congruent, $\angle A$ and $\angle D$ are right angles, and $\angle B$, $\angle C$, $\angle E$, and $\angle F$ are congruent. The area of the hexagonal region is $2116(\sqrt{2}+1)$. Find AB.

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2 The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.

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3 Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

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4 Let $(a_1, a_2, a_3, ..., a_{12})$ be a permutation of $(1, 2, 3, ..., 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \text{ and } a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

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5 When rolling a certain unfair six-sided die with faces numbered $1, 2, 3, 4, 5$, and 6, the probability of obtaining face F is greater than $\frac{1}{6}$, the probability of obtaining the face opposite is less than $\frac{1}{6}$, the probability of obtaining any one of the other four faces is $\frac{1}{6}$, and the sum of the numbers on opposite faces is 7. When two such dice are rolled, the probability of obtaining a sum of 7 is $\frac{47}{288}$. Given that the probability of obtaining face F is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

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6 Square $ABCD$ has sides of length 1. Points E and F are on $\overline{BC}$ and $\overline{CD}$, respectively, so that $\triangle AEF$ is equilateral. A square with vertex B has sides that are parallel to those of $ABCD$ and a vertex on $\overline{AE}$. The length of a side of this smaller square is $\dfrac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. Find $a+b+c$.

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7 Find the number of ordered pairs of positive integers (a, b) such that $a+b=1000$ and neither a nor b has a zero digit.

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8 There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color.

Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles may be formed?

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9 Circles $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ have their centers at (0,0), (12,0), and (24,0), and have radii 1, 2, and 4, respectively. Line t_1 is a common internal tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ and has a positive slope, and line

t_2 is a common internal tangent to $\mathcal{C}_2$ and $\mathcal{C}_3$ and has a negative slope. Given that lines t_1 and t_2 intersect at (x, y), and that $x = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

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10 Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded a point and the loser gets 0 points. The total points are accumilated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is m/n, where m and n are relatively prime positive integers. Find $m + n$.

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11 A sequence is defined as follows $a_1 = a_2 = a_3 = 1$, and, for all positive integers n, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$. Given that $a_{28} = 6090307$, $a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when $\sum_{k=1}^{28} a_k$ is divided by 1000.

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12 Equilateral $\triangle ABC$ is inscribed in a circle of radius 2. Extend $\overline{AB}$ through B to point D so that $AD = 13$, and extend $\overline{AC}$ through C to point E so that $AE = 11$. Through D, draw a line l_1 parallel to $\overline{AE}$, and through E, draw a line l_2 parallel to $\overline{AD}$. Let F be the intersection of l_1 and l_2. Let G be the point on the circle that is collinear with A and F and distinct from A. Given that the area of $\triangle CBG$ can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime, find $p + q + r$.

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13 How many integers N less than 1000 can be written as the sum of j consecutive positive odd integers from exactly 5 values of $j \geq 1$?

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14 Let S_n be the sum of the reciprocals of the non-zero digits of the integers from 1 to 10^n inclusive. Find the smallest positive integer n for which S_n is an integer.

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15 Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.
