AoPS WOOT
Art of Problem Solving

2019–2020 Syllabus

1. Induction

This class will demonstrate the fundamental problem solving technique of mathematical induction.

Example Problem: Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd. (USAMO, 2003)

Example Problem: Let (x_n) be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for $n = 1$, 2, 3, ... Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$. (Putnam, 1993)

2. Pigeonhole Principle

This class will explore uses of the Pigeonhole Principle to solve problems in discrete mathematics.

Example Problem: Prove that there exist integers a, b, and c, not all zero, and each of absolute value less than 10^6, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

(Putnam, 1980)

Example Problem: Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer. (IMO, 1985)

3. Extremal Principle

This class will demonstrate the use of the extremal principle to solve problems.

Example Problem: Given a finite number of points in the plane, not all collinear, prove that there is a straight line passing that passes through exactly two of the points.

Example Problem: On the plane are given finitely many points, such that the area of any triangle with vertices in the given points is always less than 1. Show that all these points lie inside a triangle of area 4.

4. Bijections

This class will demonstrate the use of bijections to solve certain combinatorial problems simply and effectively.

Example Problem: In how many ways can the integers from 1 to n be ordered subject to the condition that except for the first integer on the left, every integer differs by 1 from some integer to the left of it? (Putnam, 1965)

Example Problem: An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length* n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n. (USAMO, 1996)

5. Enumerative Combinatorics

This class will focus on counting problems, combinatorial identities, and other problems whose solutions involve a fair amount of algebra. Double counting proofs, the probabilistic method, and generating functions will be touched on.

Example Problem: Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a friend of the other, and each member of a hostile pair is a foe of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1-4q/n^2)$ or fewer amicable pairs. (USAMO, 1995)

Example Problem: Let p be an odd prime number. How many p-element subsets A of $\{1,2,\ldots,2p\}$ are there, the sum of whose elements is divisible by p? (IMO, 1995)

6. Angle Chasing and Cyclic Quadrilaterals

This class will cover the basics of solving geometry problems via synthetic methods with a particular focus on angle chasing and cyclic quadrilaterals.

Example Problem: In $\triangle ABC$, $AB=3$, $BC=4$, and $CA=5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF=DF$ and $\frac{DG}{EG}=\frac{3}{4}$, find length DE.

Example Problem: Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

7. Complex Numbers in Geometry A

8. Complex Numbers in Geometry B

These classes will show how to apply complex numbers to problems in geometry.

Example Problem: The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab. (AIME, 1994)

Example Problem: Let $ABCDE$ be a cyclic pentagon inscribed in a circle of center O which has angles $\angle B=120°$, $\angle C=120°$, $\angle D=130°$, $\angle E=100°$. Show that the diagonals BD and CE meet at a point belonging to the diameter of $\odot O$ with A as an endpoint. (Romania, 2002)

9. Projective Geometry

This class will cover some very basic ideas in projective geometry, like harmonics and polars, with an emphasis on solving problems via these tools.

Example Problem: Let ABC be a triangle with orthocenter H. The tangent lines from A to the circle with diameter BC touch this circle at P and Q. Prove that H, P, and Q are collinear.

Example Problem: Point M lies on diagonal BD of parallelogram $ABCD$. Line AM intersects side CD and line BC at points K and N, respectively. Let C_1 be the circle with center M and radius MA and C_2 be the circumcircle of triangle KCN. C_1, C_2 intersect at P and Q. Prove that MP and MQ are tangent to C_2.

10. Sequences & Series A

11. Sequences & Series B

These classes will cover various sequences and series, including linearly recurrent sequences and telescoping sums.

Example Problem: A sequence of numbers a_1, a_2, a_3, $\ldots$ satisfies $a_1 = 1/2$ and

$$a_1 + a_2 + \cdots + a_n = n^2 a_n$$

for all $n \geq 1$. Determine the value of a_n. (Canada, 1975)

Example Problem: Prove

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

(USAMO, 1992)

12. Inequalities A

13. Inequalities B

These classes will cover techniques for solving and deriving inequalities, as well as utilizing inequalities in other problems. Classical theorems we will cover include the AM-GM inequality, the Cauchy-Schwarz Inequality, and Jensen's Inequality. We will also discuss topics tangentially related to inequalities such as equality case analysis and convexity, which are useful in many other subjects.

Example Problem: Let a, b, c be real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

(IMO, 1995)

Example Problem: Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots. (IMO Shortlist, 2017)

14. Divisibility

This class will cover number theory problems based on divisibility at both the AIME and Olympiad level.

Example Problem: Consider the sequence a_1, a_2, $\dots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \dots).$$

Determine all positive integers that are relatively prime to every term of the sequence. (IMO, 2005)

Example Problem: Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer. (IMO, 2003)

15. Integer Polynomials A

16. Integer Polynomials B

These classes will discuss polynomials with integer coefficients and their various properties. Irreducibility will be covered in the B class.

Example Problem: Let p be a prime number and f an integer coefficient polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p - 1$. (IMO Shortlist, 1997)

Example Problem: Let a_1, a_2, $\dots$, a_n be distinct integers. Prove that the polynomial

$$(x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

is irreducible over the integers.

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