2021–2022 Syllabus

1. Symmetry and Polynomial Division

2. Roots of Polynomials

These two classes will cover the theory of polynomials. Polynomials A will include factorization, Vieta's formulas, and elementary symmetric polynomials. Polynomials B will include finite differences, the Identity Theorem and the Lagrange Interpolation Formula.

Example Problem: Factor the polynomial $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Example Problem: If $P(x)$ denotes a polynomial of degree n such that $P(k)=k/(k+1)$ for $k=0$, 1, 2, $\dots$, n, determine $P(n+1)$. (USAMO, 1975)

3. Functional Equations A

4. Functional Equations B

These classes will demonstrate techniques to solve functional equation problems.

Example Problem: Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2-y^2)=xf(x)-yf(y)$$

for all pairs of real numbers x and y. (USAMO, 2002)

Example Problem: Let $F(x)$ be a real valued function defined for all real x except for $x=0$ and $x=1$ and satisfying the functional equation

$$F(x)+F\left(\frac{x-1}{x}\right)=1+x.$$

Find all functions $F(x)$ satisfying these conditions. (Putnam, 1971)

5. Similar Triangles & Power of a Point

This class will highlight the use of similar triangles and Power of a Point, two simple but powerful techniques in Euclidean geometry.

Example Problem: Let $ABCD$ be a cyclic quadrilateral with $AB=6$, $BC=9$, and $CD=8$. Diagonals AC and BD meet at M, such that $BM=2DM$. Find AD.

Example Problem: Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q, and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle, then the center of this circles lies on line XY. (USAMO, 2009)

6. Vectors

This class will show how to apply vectors to geometry problems.

Example Problem: In tetrahedron $ABCD$, let E be on AB such that $AE : EB = 1 : 2$, let H be on BC such that $BH : HC = 1 : 2$, and let $K = AH \cap CE$. Let M be the midpoint of DK. Let HM intersect AD at L. Show that $AL : LD = 7 : 4$.

Example Problem: Let I be the incenter of triangle ABC. Prove that for any point P,

$$a \cdot PA^2 + b \cdot PB^2 + c \cdot PC^2 = a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2 + (a + b + c) \cdot IP^2.$$

(IMO Short List, 1988)

7. Geometry of the Triangle

This class will present fundamental results on the geometry of triangle, including the major centers of the triangle. There will be an emphasis on trigonometric relations.

Example Problem: A line bisects both the perimeter and area of a triangle. Prove that the line passes through the incenter of the triangle.

Example Problem: Given triangle ABC, show that the circumcenter lies on the incircle if and only if $\cos A + \cos B + \cos C = \sqrt{2}$.

8. Collinearity & Concurrency

This class will present problems involving concurrent lines and collinear points, and the tools to solve them, such as Ceva's theorem and Menelaus's theorem.

Example Problem: Let $ABCD$ and $AB'C'D'$ be two squares, oriented the same way. Prove that BB', CC', and DD' are concurrent.

Example Problem: Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F respectively. Let Γ, Γ_1, Γ_2, and Γ_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE, respectively. Let Γ and Γ_1 intersect at A and P, Γ and Γ_2 intersect at B and Q, and Γ and Γ_3 intersect at C and R. Prove that PD, QE, and CR are concurrent. (Canada, 2007)

9. Fundamentals of Number Theory

10. Fermat's Little Theorem and Euler's Theorem

12. Order, Quadratic Residues, and Primitive Roots

These three classes will cover the theory of modular arithmetic. The first class will focus on AIME level number theory problems. The second will introduce results such as Fermat's Little theorem and Euler's theorem. The third will cover quadratic residues, order, and primitive roots.

Example Problem: How many integers N less than 1000 can be written as the sum of j consecutive positive odd integers from exactly 5 values of $j \geq 1$? (AIME, 2006)

Example Problem: The number $85^9 - 21^9 + 6^9$ is divisible by an integer between 2000 and 3000. Compute that integer. (ARML, 1991)

Example Problem: Let a, b, and c be positive integers that are pairwise relatively prime, and that satisfy $a^2 - ab + b^2 = c^2$. Show that every prime factor of c is of the form $6k + 1$, where k is a positive integer.

11. Diophantine Equations

A *Diophantine equation* is an equation where we seek integer solutions. We will use the techniques presented in the modular arithmetic classes to solve such equations.

Example Problem: Determine all non-negative integral solutions $(n_1, n_2, \dots, n_{14})$ if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$

(USAMO, 1979)

13. Combinatorics

This class will introduce basic counting principles, such as distributions and the principle of Inclusion-Exclusion.

Example Problem: How many solutions does the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ have in positive integers? In non-negative integers?

Example Problem: A permutation $\{x_1, x_2, \dots, x_{2n}\}$ of the set $\{1, 2, \dots, 2n\}$ where n is a positive integer, is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \dots, 2n-1\}$. Show that, for each n, there are more permutations with property T than without. (IMO, 1989)

14. Coloring Arguments

This class will demonstrate the use of coloring arguments to solve problems in combinatorics.

Example Problem: Two opposite squares of an 8×8 chessboard are removed. Can the remaining chessboard be tiled by dominoes?

Example Problem: An 8×8 chessboard is tiled by 21 1×3 rectangles and a single 1×1 square. What are the possible locations of the 1×1 square?

15. Graph Theory

This class will cover problems and results from graph theory, a common topic in olympiad problems and one that can also be applied to many other situations as well.

Example Problem: Prove that in any polyhedron, there exists a face with at most five sides.

Example Problem: A company of $2n + 1$ people has the property that for each group of n people, there is a person among the other $n+1$ who knows everybody in that group. Here, X knows Y if and only if Y knows X. Prove that some person in the company knows everybody else. (Russia, 2001)

16. Challenging Problems in Combinatorics

This class will cover olympiad-level problems based on concepts introduced in the three previous combinatorics classes.

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