

**Mexico National Olympiad 2019**

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– Day 1

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- 1 An integer number  $m \geq 1$  is *mexica* if it's of the form  $n^{d(n)}$ , where  $n$  is a positive integer and  $d(n)$  is the number of positive integers which divide  $n$ . Find all mexica numbers less than 2019.

Note. The divisors of  $n$  include 1 and  $n$ ; for example,  $d(12) = 6$ , since 1, 2, 3, 4, 6, 12 are all the positive divisors of 12.

*Proposed by Cuauhtémoc Gómez*

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- 2 Let  $H$  be the orthocenter of acute-angled triangle  $ABC$  and  $M$  be the midpoint of  $AH$ . Line  $BH$  cuts  $AC$  at  $D$ . Consider point  $E$  such that  $BC$  is the perpendicular bisector of  $DE$ . Segments  $CM$  and  $AE$  intersect at  $F$ . Show that  $BF$  is perpendicular to  $CM$ .

*Proposed by Germán Puga*

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- 3 Let  $n \geq 2$  be an integer. Consider  $2n$  points around a circle. Each vertex has been tagged with one integer from 1 to  $n$ , inclusive, and each one of these integers has been used exactly two times. Isabel divides the points into  $n$  pairs, and draws the segments joining them, with the condition that the segments do not intersect. Then, she assigns to each segment the greatest integer between its endpoints.

a) Show that, no matter how the points have been tagged, Isabel can always choose the pairs in such a way that she uses exactly  $\lceil n/2 \rceil$  numbers to tag the segments.

b) Can the points be tagged in such a way that, no matter how Isabel divides the points into pairs, she always uses exactly  $\lceil n/2 \rceil$  numbers to tag the segments?

Note. For each real number  $x$ ,  $\lceil x \rceil$  denotes the least integer greater than or equal to  $x$ . For example,  $\lceil 3.6 \rceil = 4$  and  $\lceil 2 \rceil = 2$ .

*Proposed by Victor Domínguez*

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– Day 2

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- 4 A list of positive integers is called good if the maximum element of the list appears exactly once. A sublist is a list formed by one or more consecutive elements of a list. For example, the list 10, 34, 34, 22, 30, 22 the sublist 22, 30, 22 is good and 10, 34, 34, 22 is not. A list is very good if all its sublists are good. Find the minimum value of  $k$  such that there exists a very good list of length 2019 with  $k$  different values on it.
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5 Let  $a > b$  be relatively prime positive integers. A grasshopper stands at point 0 in a number line. Each minute, the grasshopper jumps according to the following rules:

- If the current minute is a multiple of  $a$  and not a multiple of  $b$ , it jumps  $a$  units forward.
- If the current minute is a multiple of  $b$  and not a multiple of  $a$ , it jumps  $b$  units backward.
- If the current minute is both a multiple of  $b$  and a multiple of  $a$ , it jumps  $a - b$  units forward.
- If the current minute is neither a multiple of  $a$  nor a multiple of  $b$ , it doesn't move.

Find all positions on the number line that the grasshopper will eventually reach.

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6 Let  $ABC$  be a triangle such that  $\angle BAC = 45^\circ$ . Let  $H, O$  be the orthocenter and circumcenter of  $ABC$ , respectively. Let  $\omega$  be the circumcircle of  $ABC$  and  $P$  the point on  $\omega$  such that the circumcircle of  $PBH$  is tangent to  $BC$ . Let  $X$  and  $Y$  be the circumcenters of  $PHB$  and  $PHC$  respectively. Let  $O_1, O_2$  be the circumcenters of  $PXO$  and  $PYO$  respectively. Prove that  $O_1$  and  $O_2$  lie on  $AB$  and  $AC$ , respectively.

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