**Brazil National Olympiad 2019**

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**Day 1** Monday, November 11

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**1** Let $\omega_1$ and $\omega_2$ be two circles with centers $C_1$ and $C_2$, respectively, which intersect at two points $P$ and $Q$. Suppose that the circumcircle of triangle $PC_1C_2$ intersects $\omega_1$ at $A \neq P$ and $\omega_2$ at $B \neq P$. Suppose further that $Q$ is inside the triangle $PAB$. Show that $Q$ is the incenter of triangle $PAB$.

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**2** Given are the real line and the two unique marked points $0$ and $1$. We can perform as many times as we want the following operation: we take two already marked points $a$ and $b$ and mark the reflection of $a$ over $b$. Let $f(n)$ be the minimum number of operations needed to mark on the real line the number $n$ (which is the number at a distance $|n|$ from $0$ and it is on the right of $0$ if $n > 0$ and on the left of $0$ if $n < 0$). For example, $f(0) = f(1) = 0$ and $f(-1) = f(2) = 1$. Find $f(n)$.

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**3** Let $\mathbb{R}_{>0}$ be the set of the positive real numbers. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f(xy + f(x)) = f(f(x)f(y)) + x$$

for all positive real numbers $x$ and $y$.

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**Day 2** Tuesday, November 12

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**4** Prove that for every positive integer $m$ there exists a positive integer $n_m$ such that for every positive integer $n \geq n_m$, there exist positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{a_1^m} + \frac{1}{a_2^m} + \ldots + \frac{1}{a_n^m} = 1.$$

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**5** (a) Prove that given constants $a, b$ with $1 < a < 2 < b$, there is no partition of the set of positive integers into two subsets $A_0$ and $A_1$ such that: if $j \in \{0, 1\}$ and $m, n$ are in $A_j$, then either $n/m < a$ or $n/m > b$.

(b) Find all pairs of real numbers $(a, b)$ with $1 < a < 2 < b$ for which the following property holds: there exists a partition of the set of positive integers into three subsets $A_0, A_1, A_2$ such that if $j \in \{0, 1, 2\}$ and $m, n$ are in $A_j$, then either $n/m < a$ or $n/m > b$.

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**6** Let $A_1A_2A_3A_4A_5$ be a convex, cyclic pentagon with $\angle A_i + \angle A_{i+1} > 180^\circ$ for all $i \in \{1, 2, 3, 4, 5\}$ (all indices modulo $5$ in the problem). Define $B_i$ as the intersection of lines $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$, forming a star. The circumcircles of triangles $A_{i-1}B_{i-1}A_i$ and $A_iB_iA_{i+1}$ meet again at $C_i \neq A_i$, and the circumcircles of triangles $B_{i-1}A_iB_i$ and $B_iA_{i+1}B_{i+1}$ meet again at $D_i \neq B_i$. Prove that the ten lines $A_iC_i, B_iD_i$, $i \in \{1, 2, 3, 4, 5\}$, have a common point.

**Level 2** Level 2

**1** An eight-digit number is said to be 'robust' if it meets both of the following conditions:
(i) None of its digits is $0$.
(ii) The difference between two consecutive digits is $4$ or $5$.

Answer the following questions:

(a) How many are robust numbers?
(b) A robust number is said to be 'super robust' if all of its digits are distinct. Calculate the sum of all
the super robust numbers.

**2** Let $a, b$ and $k$ be positive integers with $k > 1$ such that $lcm(a, b) + gcd(a, b) = k(a + b)$.
Prove that $a + b \geq 4k$

**3** Let $ABC$ be an acutangle triangle inscribed in a circle $\Gamma$ of center $O$. Let $D$ be the height of the vertex $A$. Let E and F be points over $\Gamma$ such that $AE = AD = AF$. Let $P$ and $Q$ be the intersection points of the $EF$ with sides $AB$ and $AC$ respectively. Let $X$ be the second intersection point of $\Gamma$ with the circle circumscribed to the triangle $APQ$. Show that the lines $XD$ and $AO$ meet at a point above sobre

**4** Let $ABC$ be an acutangle triangle and $D$ any point on the $BC$ side. Let $E$ be the symmetrical of $D$ in $AC$ and $F$ is the symmetrical $D$ relative to $AB$. $A$ straight $ED$ intersects straight $AB$ at $G$, while straight $FD$ intersects the line $AC$ in $H$. Prove that the points $A, E, F, G$ and $H$ are on the same circumference.

**5** In the picture below, a white square is surrounded by four black squares and three white squares. They are surrounded by seven black squares.
`https://i.stack.imgur.com/Dalmm.png`
What is the maximum number of white squares that can be surrounded by $n$ black squares?

**6** In the Cartesian plane, all points with both integer coordinates are painted blue. Blue colon they are said to be **mutually visible** if the line segment connecting them has no other blue dots. Prove that
There is a set of $2019$ blue dots that are mutually visible two by two.