Korea National Olympiad 2019

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–	Day 1
1	The sequence $a_1, a_2, ..., a_{2019}$ satisfies the following condition. $a_1 = 1, a_{n+1} = 2019a_n + 1$ Now let $x_1, x_2, ..., x_{2019}$ real numbers such that $x_1 = a_{2019}, x_{2019} = a_1$ (The others are arbitary.) Prove that $\sum_{k=1}^{2018}(x_{k+1} - 2019x_k - 1)^2 \geq \sum_{k=1}^{2018}(a_{2019-k} - 2019a_{2020-k} - 1)^2$
2	Triangle ABC is an scalene triangle. Let I the incenter, Ω the circumcircle, E the A-excenter of triangle ABC. Let Γ the circle centered at E and passes A. Γ and Ω intersect at point $D(\neq A)$, and the perpendicular line of BC which passes A meets Γ at point $K(\neq A)$. L is the perpendicular foot from I to AC. Now if AE and DK intersects at F, prove that $BE \cdot CI = 2 \cdot CF \cdot CL$.
3	Suppose that positive integers m, n, k satisfy the equations $$m^2 + 1 = 2n^2, 2m^2 + 1 = 11k^2.$$ Find the residue when n is divided by 17.
4	Let $(x_1, y_1, z_1), (x_2, y_2, z_2), \cdots, (x_{19}, y_{19}, z_{19})$ be integers. Prove that there exist pairwise distinct subscripts i, j, k such that $x_i + x_j + x_k$, $y_i + y_j + y_k$, $z_i + z_j + z_k$ are all multiples of 3.
–	Day 2
5	Find all functions f such that $f : \mathbb{R} \to \mathbb{R}$ and $f(f(x) - x + y^2) = yf(y)$
6	In acute triangle ABC, $AB > AC$. Let I the incenter, Ω the circumcircle of triangle ABC, and D the foot of perpendicular from A to BC. AI intersects Ω at point $M(\neq A)$, and the line which passes M and perpendicular to AM intersects AD at point E. Now let F the foot of perpendicular from I to AD. Prove that $ID \cdot AM = IE \cdot AF$.
7	For prime $p \equiv 1 \pmod 7$, prove that there exists some positive integer m such that $m^3 + m^2 - 2m - 1$ is a multiple of p.
8	There are two countries A and B, where each countries have $n(\geq 2)$ airports. There are some two-way flights among airports of A and B, so that each airport has exactly 3 flights. There might be multiple flights among two airports; and there are no flights among airports of the same country. A travel agency wants to plan an *exotic traveling course* which travels through all

$2n$ airports exactly once, and returns to the initial airport. If N denotes the number of all exotic traveling courses, then prove that $\frac{N}{4n}$ is an even integer.

(Here, note that two exotic traveling courses are different if their starting place are different.)
