Brazil Undergrad MO 2016

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Day 1 Day 1

1 Let $(a_n)_{n\geq 1}$ s sequence of reals such that $\sum_{n\geq 1} \frac{a_n}{n}$ converges. Show that $\lim_{n\to\infty} \frac{1}{n}\cdot\sum_{k=1}^{n} a_k = 0$

2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2+y^2f(x)) = xf(y)^2 - f(x)^2$$

for every $x, y \in \mathbb{R}$

3 Let it $k \geq 1$ be an integer. Define the sequence $(a_n)_{n\geq 1}$ by $a_0 = 0, a_1 = 1$ and

$$a_{n+2} = ka_{n+1} + a_n$$

for $n \geq 0$.
Let it p an odd prime number.
Denote $m(p)$ as the smallest positive integer m such that $p|a_m$.
Denote $T(p)$ as the smallest positive integer T such that for every natural j we gave $p|(a_{T+j} - a_j)$.

- Show that $T(p) \leq (p-1)\cdot m(p)$.
- Show that if $T(p) = (p-1)\cdot m(p)$ then

$$\prod_{1\leq j\leq T(p)-1}^{j\not\equiv 0 \pmod{m(p)}} a_j \equiv (-1)^{m(p)-1} \pmod{p}$$

Day 2 Day 2

4 Let

$$A = \begin{pmatrix} 4 & -\sqrt{5} \\ 2\sqrt{5} & -3 \end{pmatrix}$$

Find all pairs of integers m, n with $n \geq 1$ and $|m| \leq n$ such as all entries of $A^n - (m + n^2)A$ are integer.

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5 A soccer ball is usually made from a polyhedral fugure model, with two types of faces, hexagons and pentagons, and in every vertex incide three faces - two hexagons and one pentagon.

We call a polyhedron *soccer-ball* if it is similar to the traditional soccer ball, in the following sense: its faces are m-gons or n-gons, $m \neq n$, and in every vertex incide three faces, two of them being m-gons and the other one being an n-gon.

- Show that m needs to be even.
- Find all soccer-ball polyhedra.

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6 Let it $C, D > 0$. We call a function $f : \mathbb{R} \to \mathbb{R}$ *pretty* if f is a C^2-class, $|x^3 f(x)| \leq C$ and $|xf''(x)| \leq D$.

- Show that if f is pretty, then, given $\epsilon \geq 0$, there is a $x_0 \geq 0$ such that for every x with $|x| \geq x_0$, we have $|x^2 f'(x)| < \sqrt{2CD} + \epsilon$.
- Show that if $0 < E < \sqrt{2CD}$ then there is a pretty function f such that for every $x_0 \geq 0$ there is a $x > x_0$ such that $|x^2 f'(x)| > E$.
