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1 A polynomial is called positivist if it can be written as a product of two non-constant polynomials with non-negative real coefficients. Let $f(x)$ be a polynomial of degree greater than one such that $f(x^n)$ is positivist for some positive integer n. Show that $f(x)$ is positivist.

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2 Let a and b be fixed positive integers. Show that the set of primes that divide at least one of the terms of the sequence $a_n = a \cdot 2017^n + b \cdot 2016^n$ is infinite.

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3 Let $X = \{(x,y) \in \mathbb{R}^2 | y \geq 0, x^2 + y^2 = 1\} \cup \{(x,0), -1 \leq x \leq 1\}$ be the edge of the closed semicircle with radius 1.

a) Let $n > 1$ be an integer and $P_1, P_2, \dots, P_n \in X$. Show that there exists a permutation $\sigma \colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ such that

$$\sum_{j=1}^{n} |P_{\sigma(j+1)} - P_{\sigma(j)}|^2 \leq 8$$

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Where $\sigma(n+1) = \sigma(1)$.

b) Find all sets $\{P_1, P_2, \dots, P_n\} \subset X$ such that for any permutation $\sigma \colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$,

$$\sum_{j=1}^{n} |P_{\sigma(j+1)} - P_{\sigma(j)}|^2 \geq 8$$

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Where $\sigma(n+1) = \sigma(1)$.

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4 Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers in which $\lim_{n\to\infty} a_n = 0$ such that there is a constant $c > 0$ so that for all $n \geq 1$, $|a_{n+1} - a_n| \leq c \cdot a_n^2$. Show that exists $d > 0$ with $na_n \geq d, \forall n \geq 1$.

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5 Let $d \leq n$ be positive integers and A a real $d \times n$ matrix. Let $\sigma(A)$ be the supremum of $\inf_{v\in W, |v|=1} |Av|$ over all subspaces W of R^n with dimension d.

For each $j \leq d$, let $r(j) \in \mathbb{R}^n$ be the jth row vector of A. Show that:

$$\sigma(A) \leq \min_{i\leq d} d(r(i), \langle r(j), j \neq i \rangle) \leq \sqrt{n}\sigma(A)$$

In which all are euclidian norms and $d(r(i), \langle r(j), j \neq i \rangle)$ denotes the distance between $r(i)$ and the span of $r(j), 1 \leq j \leq d, j \neq i$.

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6 Let's consider words over the alphabet $\{a, b\}$ to be sequences of a and b with finite length. We say $u \leq v$ if u is a subword of v if we can get u erasing some letter of v (for example $aba \leq abbab$). We say that u differentiates the words x and y if $u \leq x$ but $u \not\leq y$ or vice versa.

Let m and l be positive integers. We say that two words are $m-$equivalents if there does not exist some u with length smaller than m that differentiates x and y.

a) Show that, if $2m \leq l$, there exists two distinct words with length l $m-$equivalents.
b) Show that, if $2m > l$, any two distinct words with length l aren't $m-$equivalent.
