Brazil Undergrad MO 2019

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1 Let I and 0 be the square identity and null matrices, both of size 2019. There is a square matrix A
with rational entries and size 2019 such that:
a) $A^3+6A^2-2I=0$?
b) $A^4+6A^3-2I=0$?

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3 Let a,b,c be constants and a,b,c are positive real numbers. Prove that the equations $2x+y+z=\sqrt{c^2+z^2}+\sqrt{c^2+y^2}$ $x+2y+z=\sqrt{b^2+x^2}+\sqrt{b^2+z^2}$ $x+y+2z=\sqrt{a^2+x^2}+\sqrt{a^2+y^2}$ have exactly one real solution (x,y,z) with $x,y,z\geq 0$.

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4 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for any (x,y) real numbers we have $f(xf(y)+f(x))+f(y^2)=f(x)+yf(x+y)$

Problem 5 Let $M,k>0$ integers.

Let $X(M,k)$ the (infinite) set of all integers that can be factored as ${p_1}^{e_1}\cdot{p_2}^{e_2}\cdot\ldots\cdot{p_r}^{e_r}$ where each p_i is not smaller than M and also each e_i is not smaller than k.

Let $Z(M,k,n)$ the number of elements of $X(M,k)$ not bigger than n.

Show that there are positive reals $c(M,k)$ and $\beta(M,k)$ such that

$$\lim_{n\to\infty}\frac{Z(M,k,n)}{n^{\beta(M,k)}}=c(M,k)$$

and find $\beta(M,k)$

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6 In a hidden friend, suppose no one takes oneself. We say that the hidden friend has "marmalade" if
there are two people A and B such that A took B and B took A. For each positive integer n, let $f(n)$ be the number of hidden friends with n people where there is no "marmalade", i.e. $f(n)$ is equal to the number of permutations σ of $1,2,...,n$ such that:

*$\sigma(i)\neq i$ for all $i=1,2,...,n$

* there are no $1\leq i<j\leq n$ such that $\sigma(i)=j$ and $\sigma(j)=i$.

Determine the limit

$\lim_{n\to+\infty}\frac{f(n)}{n!}$