2020 China National Olympiad

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Day 1 Nov. 25th, 2019

1 Let $a_1, a_2, \cdots, a_{41} \in \mathbb{R}$, such that $a_{41} = a_1, \sum_{i=1}^{40} a_i = 0$, and for any $i = 1, 2, \cdots, 40, |a_i - a_{i+1}| \leq 1$. Determine the greatest possible value of $(1) a_{10} + a_{20} + a_{30} + a_{40}$; $(2) a_{10} \cdot a_{20} + a_{30} \cdot a_{40}$.

2 In triangle ABC, $AB > AC$. The bisector of $\angle BAC$ meets BC at D. P is on line DA, such that A lies between P and D. PQ is tangent to $\odot(ABD)$ at Q. PR is tangent to $\odot(ACD)$ at R. CQ meets BR at K. The line parallel to BC and passing through K meets QD, AD, RD at E, L, F, respectively. Prove that $EL = KF$.

3 Let S be a set, $|S| = 35$. A set F of mappings from S to itself is called to be satisfying property $P(k)$, if for any $x, y \in S$, there exist $f_1, \cdots, f_k \in F$ (not necessarily different), such that $f_k(f_{k-1}(\cdots(f_1(x)))) = f_k(f_{k-1}(\cdots(f_1(y))))$.
Find the least positive integer m, such that if F satisfies property $P(2019)$, then it also satisfies property $P(m)$.

Day 2 Nov. 26th, 2019

4 Find the largest positive constant C such that the following is satisfied: Given n arcs (containing their endpoints) $A_1, A_2, \ldots, A_n$ on the circumference of a circle, where among all sets of three arcs (A_i, A_j, A_k) $(1 \leq i < j < k \leq n)$, at least half of them has $A_i \cap A_j \cap A_k$ nonempty, then there exists $l > Cn$, such that we can choose l arcs among $A_1, A_2, \ldots, A_n$, whose intersection is nonempty.

5 Given any positive integer c, denote $p(c)$ as the largest prime factor of c. A sequence $\{a_n\}$ of positive integers satisfies $a_1 > 1$ and $a_{n+1} = a_n + p(a_n)$ for all $n \geq 1$. Prove that there must exist at least one perfect square in sequence $\{a_n\}$.

6 Does there exist positive reals $a_0, a_1, \ldots, a_{19}$, such that the polynomial $P(x) = x^{20} + a_{19}x^{19} + \ldots + a_1 x + a_0$ does not have any real roots, yet all polynomials formed from swapping any two coefficients a_i, a_j has at least one real root?

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