USA TSTST 2015

www.artofproblemsolving.com/community/c102039

by v_Enhance, raxu

– Day 1

---

1 Let $a_1, a_2, \dots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \le k \le n$ is good if there exists some ℓ with $1 \le \ell \le m$ such that $a_k + a_{k+1} + ... + a_{k+\ell-1} \ge 0$, where the indices are taken modulo n. Let T be the set of all good indices. Prove that $\sum\limits_{k \in T} a_k \ge 0$.

Proposed by Mark Sellke

---

2 Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.
(The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)

Proposed by Ivan Borsenco

---

3 Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $p_1, p_2, ..., p_k$ of M, all prime factors of $p_1p_2...p_k + 1$ are also in M. Prove that $M = P$.

Proposed by Alex Zhai

– Day 2

---

4 Let x, y, and z be real numbers (not necessarily positive) such that $x^4 + y^4 + z^4 + xyz = 4$. Show that $x \le 2$ and $\sqrt{2-x} \ge \frac{y+z}{2}$.

Proposed by Alyazeed Basyoni

---

5 Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Proposed by Iurie Boreico

---

6 A *Nim-style game* is defined as follows. Two positive integers k and n are specified, along with a finite set S of k-tuples of integers (not necessarily positive). At the start of the game, the

k-tuple $(n, 0, 0, ..., 0)$ is written on the blackboard.
A legal move consists of erasing the tuple $(a_1, a_2, ..., a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, ..., a_k + b_k)$, where $(b_1, b_2, ..., b_k)$ is an element of the set S. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.
Prove that there is a choice of k and S with the following property: the first player has a winning strategy if n is a power of 2, and otherwise the second player has a winning strategy.

Proposed by Linus Hamilton
