Math Prize for Girls Olympiad 2019

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1 Let A_1, A_2, $\dots$, A_n be finite sets. Prove that

$$\Big| \bigcup_{1 \le i \le n} A_i \Big| \ge \frac{1}{2} \sum_{1 \le i \le n} |A_i| - \frac{1}{6} \sum_{1 \le i < j \le n} |A_i \cap A_j| \; .$$

Recall that if S is a finite set, then its cardinality $|S|$ is the number of elements of S.

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2 Let ABC be an equilateral triangle with side length 1. Say that a point X on side $\overline{BC}$ is *balanced* if there exists a point Y on side $\overline{AC}$ and a point Z on side $\overline{AB}$ such that the triangle XYZ is a right isosceles triangle with $XY = XZ$. Find with proof the length of the set of all balanced points on side $\overline{BC}$.

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3 Say that a positive integer is *red* if it is of the form n^{2020}, where n is a positive integer. Say that a positive integer is *blue* if it is not red and is of the form n^{2019}, where n is a positive integer. True or false: Between every two different red positive integers greater than $10^{100,000,000}$, there are at least 2019 blue positive integers. Prove that your answer is correct.

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4 Let n be a positive integer. Let d be an integer such that $d \ge n$ and d is a divisor of $\frac{n(n+1)}{2}$. Prove that the set $\{1, 2, \dots, n\}$ can be partitioned into disjoint subsets such that the sum of the numbers in each subset equals d.
