2020 Hong Kong (China) Mathematical Olympiad

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by Blastzit

1 Given that a_n and b_n are two sequences of integers defined by

$$a_1=1, a_2=10, a_{n+1}=2a_n+3a_{n-1} \quad \text{for } n=2,3,4,\ldots,$$
$$b_1=1, b_2=8, b_{n+1}=3b_n+4b_{n-1} \quad \text{for } n=2,3,4,\ldots.$$

Prove that, besides the number 1, no two numbers in the sequences are identical.

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2 Let $S=1,2,\ldots,100$. Consider a partition of S into $S_1,S_2,\ldots,S_n$ for some n, i.e. S_i are nonempty, pairwise disjoint and $S=\bigcup_{i=1}^{n} S_i$. Let a_i be the average of elements of the set S_i. Define the score of this partition by

$$\frac{a_1+a_2+\ldots+a_n}{n}.$$

Among all n and partitions of S, determine the minimum possible score.

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3 Let ΔABC be an isosceles triangle with $AB=AC$. The incircle Γ of ΔABC has centre I, and it is tangent to the sides AB and AC at F and E respectively. Let Ω be the circumcircle of ΔAFE. The two external common tangents of Γ and Ω intersect at a point P. If one of these external common tangents is parallel to AC, prove that $\angle PBI=90^\circ$.

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4 There are $n\geq 3$ cities in a country and between any two cities A and B, there is either a one way road from A to B, or a one way road from B to A (but never both). Assume the roads are built such that it is possible to get from any city to any other city through these roads, and define $d(A,B)$ to be the minimum number of roads you must go through to go from city A to B. Consider all possible ways to build the roads. Find the minimum possible average value of $d(A,B)$ over all possible ordered pairs of distinct cities in the country.
