Brazil Undergrad MO - Galois-Noether 2018 #1 (Phase 1)

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by Zelderis

1	An equilateral triangle is cut as shown in figure 1 and the parts are used to form figure 2. What is the shape of figure 2?
2	Let $f, g : \mathbb{R} \to \mathbb{R}$ function such that $f(x + g(y)) = -x + y + 1$ for each pair of real numbers x e y. What is the value of $g(x + f(y)$?
3	How many permutations a_1, a_2, a_3, a_4 of $1, 2, 3, 4$ satisfy the condition that for $k = 1, 2, 3$, the list $a_1, ..., a_k$ contains a number greater than k?
4	Consider the property that each a element of a group G satisfies $a^2 = e$, where e is the identity element of the group. Which of the following statements is not always valid for a group G with this property? (a) G is commutative (b) G has infinite or even order (c) G is Noetherian (d) G is vector space over $\mathbb{Z}_2$
5	Consider the set $A = \left\{ \frac{j}{4} + \frac{100}{j} \mid j = 1, 2, 3, .. \right\}$ What is the smallest number that belongs to the A set?
6	Given an equilateral triangle ABC in the plane, how many points P in the plane are such that the three triangles APB, BPC and CPA are isosceles and not degenerate?
7	Unless of isomorphisms, how many simple four-vertex graphs are there?
8	A student will take an exam in which they have to solve three chosen problems by chance of a list of 10 possible problems. It will be approved if it correctly resolves two problems. Considering that the student can solve five of the problems on the list and not know how to solve others, how likely is he to pass the exam?
9	How many functions $f : \{1, 2, 3\} \to \{1, 2, 3\}$ satisfy $f(f(x)) = f(f(f(x)))$ for every x?
10	How many ordered pairs of real numbers (a, b) satisfy equality $\lim_{x \to 0} \frac{\sin^2 x}{e^{ax} - 2bx - 1} = \frac{1}{2}$?
12	Let ABC be an equilateral triangle. A point P is chosen at random within this triangle. What is the probability that the sum of the distances from point P to the sides of triangle ABC are

measures of the sides of a triangle?

13 A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x)f(f(x)) = 1$ for every real x and $f(2020) = 2019$. What is the value of $f(2018)$?

14 What is the arithmetic mean of all values of the expression $|a_1 - a_2| + |a_3 - a_4|$
Where a_1, a_2, a_3, a_4 is a permutation of the elements of the set $1, 2, 3, 4$?

15 A real number to is randomly and uniformly chosen from the $[-3, 4]$ interval. What is the probability that all roots of the polynomial $x^3 + ax^2 + ax + 1$ are real?

16 A positive integer of at least two digits written in the base 10 is called 'ascending' if the digits increase in value from left to right. For example, 123 is 'ascending', but 132 and 122 is not. How many 'ascending' numbers are there?

17 In the figure, a semicircle is folded along the AN string and intersects the MN diameter in B. $MB : BN = 2 : 3$ and $MN = 10$ are known to be. If $AN = x$, what is the value of x^2?

19 What is the largest amount of complex z solutions a system can have? $|z - 1||z + 1| = 1$ $Im(z) = b$?
(where b is a real constant)

20 What is the largest number of points that can exist on a plane so that each distance between any two of them is an odd integer?

21 Consider $p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + 1$ a polynomial of positive real coefficients, degree $n \geq 2$ e with n real roots. Which of the following statements is always true?

a) $p(2) < 2(2^{n-1} + 1)$ (b) $p(1) < 3$ c) $p(1) > 2^n$ d) $p(3) < 3(2^{n-1} - 2)$

22 What is the value of the improper integral $\int_0^\pi \log(\sin(x))dx$?

23 How many prime numbers p the number $p^3 - 4p + 9$ is a perfect square

24 What is the value of the series $\sum_{1 \leq l < m < n} \frac{1}{5^l 3^m 2^n}$

25 Consider the $\mathbb{Z}/(10)$ additive group automorphism group of integers module 10, that is, $A = \{\phi : \mathbb{Z}/(10) \to \mathbb{Z}/(10) | \phi - automorphism\}$