National Olympiad Second Round 2019

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Day 1 December 21st, 2019

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1 a, b, c are positive real numbers such that

$$(\sqrt{ab}-1)(\sqrt{bc}-1)(\sqrt{ca}-1)=1$$

At most, how many of the numbers:

$$a-\frac{b}{c}, a-\frac{c}{b}, b-\frac{a}{c}, b-\frac{c}{a}, c-\frac{a}{b}, c-\frac{b}{a}$$

can be bigger than 1?

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2 Let $d(n)$ denote the number of divisors of a positive integer n. If k is a given odd number, prove that there exist an increasing arithmetic progression in positive integers $(a_1, a_2, \dots a_{2019})$ such that $gcd(k, d(a_1)d(a_2)\dots d(a_{2019}))=1$

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3 There are 2019 students in a school, and some of these students are members of different student clubs. Each student club has an advisory board consisting of 12 students who are members of that particular club. An *advisory meeting* (for a particular club) can be realized only when each participant is a member of that club, and moreover, each of the 12 students forming the advisory board are present among the participants. It is known that each subset of at least 12 students in this school can realize an advisory meeting for exactly one student club. Determine all possible numbers of different student clubs with exactly 27 members.

Day 2 December 22nd, 2019

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4 In a triangle ΔABC, $|AB|=|AC|$. Let M be on the minor arc AC of the circumcircle of ΔABC different than A and C. Let BM and AC meet at E and the bisector of $\angle BMC$ and BC meet at F such that $\angle AFB = \angle CFE$. Prove that the triangle ΔABC is equilateral.

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5 Let $f : \{1, 2, \dots, 2019\} \to \{-1, 1\}$ be a function, such that for every $k \in \{1, 2, \dots, 2019\}$, there exists an $\ell \in \{1, 2, \dots, 2019\}$ such that

$$\sum_{i\in\mathbb{Z}:(\ell-i)(i-k)\geqslant 0} f(i) \leqslant 0.$$

Determine the maximum possible value of

$$\sum_{i\in\mathbb{Z}:1\leqslant i\leqslant 2019} f(i).$$

6 Given an integer $n > 2$ and an integer a, if there exists an integer d such that $n \mid a^d - 1$ and $n \nmid a^{d-1} + \cdots + 1$, we say [i]$a$ is $n-$separating[/i]. Given any n¿2, let the [i]defect of n[/i] be defined as the number of integers a such that $0 < a < n$, $(a, n) = 1$, and a is not [i] $n-$separating[/i]. Determine all integers $n > 2$ whose defect is equal to the smallest possible value.