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– Individual Competition, Day 1

**1** Let $M(n) = \{n, n+1, n+2, n+3, n+4, n+5\}$ be a set of 6 consecutive integers. Let's take all values of the form

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$$

with the set $\{a, b, c, d, e, f\} = M(n)$.
Let

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = \frac{xvw + yuw + zuv}{uvw}$$

be the greatest of all these values.

a) show: for all odd $n$ hold: $\gcd(xvw+yuw+zuv, uvw) = 1$ iff $\gcd(x, u) = \gcd(y, v) = \gcd(z, w) = 1$.
b) for which positive integers $n$ hold $\gcd(xvw + yuw + zuv, uvw) = 1$?

**2** Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3 P(x-1) - (x-1)^3 P(x+1) = 4(x^2-1)P(x)$$

for all real numbers $x$.

**3** $ABCD$ is a tetrahedron.
Let $K$ be the center of the incircle of $CBD$.
Let $M$ be the center of the incircle of $ABD$.
Let $L$ be the gravycenter of $DAC$.
Let $N$ be the gravycenter of $BAC$.

Suppose $AK$, $BL$, $CM$, $DN$ have one common point.
Is $ABCD$ necessarily regular?

– Individual Competition, Day 2

**4** A positive integer $d$ is called *nice* iff for all positive integers $x, y$ hold: $d$ divides $(x+y)^5 - x^5 - y^5$ iff $d$ divides $(x+y)^7 - x^7 - y^7$ .
a) Is 29 nice?
b) Is 2006 nice?
c) Prove that infinitely many nice numbers exist.

**5** Prove that for all positive integers $n$ and all positive reals $a, b, c$ the following inequality holds:

$$\frac{a^{n+1}}{a^n+a^{n-1}b+\ldots+b^n}+\frac{b^{n+1}}{b^n+b^{n-1}c+\ldots+c^n}+\frac{c^{n+1}}{c^n+c^{n-1}a+\ldots+a^n}\geq\frac{a+b+c}{n+1}$$

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**6** Let $D$ be an interior point of the triangle $ABC$. $CD$ and $AB$ intersect at $D_c$, $BD$ and $AC$ intersect at $D_b$, $AD$ and $BC$ intersect at $D_a$.
Prove that there exists a triangle $KLM$ with orthocenter $H$ and the feet of altitudes $H_k \in LM, H_l \in KM, H_m \in KL$, so that $(AD_cD) = (KH_mH)$ $(BD_cD) = (LH_mH)$ $(BD_aD) = (LH_kH)$ $(CD_aD) = (MH_kH)$ $(CD_bD) = (MH_lH)$ $(AD_bD) = (KH_lH)$

where $(PQR)$ denotes the area of the triangle $PQR$

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**–** Team Competition

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**7** Find all nonnegative integers $m, n$ so that

$$\sum_{k=1}^{2^m}\lfloor\frac{kn}{2^m}\rfloor\in\{28,29,30\}$$

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**8** Let $A\subset\{x|0\leq x<1\}$ with the following properties:
1. $A$ has at least 4 members.
2. For all pairwise different $a, b, c, d \in A$, $ab+cd\in A$ holds.
Prove: $A$ has infinetly many members.

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**9** We have an 8x8 chessboard with 64 squares. Then we have 3x1 dominoes which cover exactly 3 squares. Such dominoes can only be moved parallel to the borders of the chessboard and also only if the passing squares are free. If no dominoes can be moved, then the position is called stable.
a. Find the smalles number of covered squares neccessary for a stable position.
b. Prove: There exist a stable position with only one square uncovered.
c. Find all Squares which are uncoverd in at least one position of b).

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**10** Let $ABCDS$ be a (not neccessarily straight) pyramid with a rectangular base $ABCD$ and acute triangular faces $ABS, BCS, CDS, DAS$. We consider all cuboids which are inscribed inside the pyramid with its base being in the plane $ABCD$ and its upper vertexes are in the triangular faces (one in each).
Find the locus of the midpoints of these cuboids.

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