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– Individual Competition, Day 1

1 Determine the number of positive integers a, so that there exist nonnegative integers $x_0, x_1, \dots, x_{2001}$ which satisfy the equation

$$a^{x_0} = \sum_{i=1}^{2001} a^{x_i}$$

2 Let n be a positive integer greater than 2. Solve in nonnegative real numbers the following system of equations

$$x_k + x_{k+1} = x_{k+2}^2 \quad , \quad k = 1, 2, \cdots, n$$

where $x_{n+1} = x_1$ and $x_{n+2} = x_2$.

3 Let a, b, c be sides of a triangle. Prove that

$$2 < \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - \frac{a^3+b^3+c^3}{abc} \leq 3$$

– Individual Competition, Day 2

4 Prove that if a, b, c, d are lengths of the successive sides of a quadrangle (not necessarily convex) with the area equal to S, then the following inequality holds

$$S \leq \frac{1}{2}(ac+bd).$$

For which quadrangles does the inequality become equality?

5 The fields of the 8×8 chessboard are numbered from 1 to 64 in the following manner. For $i = 1, 2, \cdots, 63$ the field numbered by $i+1$ can be reached from the field numbered by i by one move of the knight. Let us choose positive real numbers $x_1, x_2, \cdots, x_{64}$. For each white field numbered by i define the number $y_i = 1 + x_i^2 - \sqrt[3]{x_{i-1}^2 x_{i+1}}$ and for each black field numbered by j define the number $y_j = 1 + x_j^2 - \sqrt[3]{x_{j-1} x_{j+1}^2}$ where $x_0 = x_{64}$ and $x_1 = x_{65}$. Prove that

$$\sum_{i=1}^{64} y_i \geq 48$$

6 Let k be a fixed positive integer. Consider the sequence definited by

$$a_0=1 \quad ,a_{n+1}=a_n+\lfloor \sqrt[k]{a_n} \rfloor \quad ,n=0,1,\cdots$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For each k find the set A_k containing all integer values of the sequence $(\sqrt[k]{a_n})_{n\geq 0}$.

– Team Competition

7 Consider the set A containing all positive integers whose decimal expansion contains no 0, and whose sum $S(N)$ of the digits divides N.

(a) Prove that there exist infinitely many elements in A whose decimal expansion contains each digit the same number of times as each other digit.

(b) Explain that for each positive integer k there exist an element in A having exactly k digits.

8 The prism with the regular octagonal base and with all edges of the length equal to 1 is given. The points $M_1,M_2,\cdots,M_{10}$ are the midpoints of all the faces of the prism. For the point P from the inside of the prism denote by P_i the intersection point (not equal to M_i) of the line M_iP with the surface of the prism. Assume that the point P is so chosen that all associated with P points P_i do not belong to any edge of the prism and on each face lies exactly one point P_i. Prove that

$$\sum_{i=1}^{10}\frac{M_iP}{M_iP_i}=5$$

9 Let A be a set with $2n$ elements, and let $A_1, A_2 ..., A_m$ be subsets of Ae ach one with n elements. Find the greatest possible m, such that it is possible to select these m subsets in such a way that the intersection of any 3 of them has at most one element.

10 The sequence $a_1,a_2,\cdots,a_{2010}$ has the following properties:

(1) each sum of the 20 successive values of the sequence is nonnegative,

(2) $|a_ia_{i+1}|\leq 1$ for $i=1,2,\cdots,2009$.

Determine the maximal value of the expression $\sum_{i=1}^{2010}a_i$.

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