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– Individual Competition, Day 1, June 24th 2004

1 Let $S(n)$ be the sum of digits for any positive integer n (in decimal notation).
Let $N = \sum_{k=10^{2003}}^{10^{2004}-1} S(k)$. Determine $S(N)$.

2 In a triangle ABC let D be the intersection of the angle bisector of γ, angle at C, with the side AB. And let F be the area of the triangle ABC. Prove the following inequality:

$$2 \cdot \ F \cdot \left(\frac{1}{AD} - \frac{1}{BD}\right) \leq AB.$$

3 Solve the following system of equations in $\mathbb{R}$ where all square roots are non-negative:
$a - \sqrt{1-b^2} + \sqrt{1-c^2} = d$
$b - \sqrt{1-c^2} + \sqrt{1-d^2} = a$
$c - \sqrt{1-d^2} + \sqrt{1-a^2} = b$
$d - \sqrt{1-a^2} + \sqrt{1-b^2} = c$

– Individual Competition, Day 2, June 25th 2004

4 Determine all $n \in \mathbb{N}$ for which $n^{10} + n^5 + 1$ is prime.

5 Determine all n for which the system with of equations can be solved in $\mathbb{R}$:

$$\sum_{k=1}^{n} x_k = 27$$

and

$$\prod_{k=1}^{n} x_k = \left(\frac{3}{2}\right)^{24}.$$

6 For $n = 2^m$ (m is a positive integer) consider the set $M(n) = \{1, 2, ..., n\}$ of natural numbers. Prove that there exists an order $a_1, a_2, ..., a_n$ of the elements of M(n), so that for all $1 \leq i < j < k \leq n$ holds: $a_j - a_i \neq a_k - a_j$.

– Team Competition, June 26th 2004

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7 Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}$ which satisfy the following condition for all pairs (x, y) of *relatively prime* positive integers:

$$f(x + y) = f(x + 1) + f(y + 1).$$

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8 a.) Prove that for $n = 4$ or $n \geq 6$ each triangle ABC can be decomposed in n similar (not necessarily congruent) triangles.

b.) Show: An equilateral triangle can neither be composed in 3 nor 5 triangles.

c.) Is there a triangle ABC which can be decomposed in 3 and 5 triangles, analogously to a.). Either give an example or prove that there is not such a triangle.

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9 Given are the sequences

$$(..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...); (..., b_{-2}, b_{-1}, b_0, b_1, b_2, ...); (..., c_{-2}, c_{-1}, c_0, c_1, c_2, ...)$$

of positive real numbers. For each integer n the following inequalities hold:

$$a_n \geq \frac{1}{2}(b_{n+1} + c_{n-1})$$

$$b_n \geq \frac{1}{2}(c_{n+1} + a_{n-1})$$

$$c_n \geq \frac{1}{2}(a_{n+1} + b_{n-1})$$

Determine a_{2005}, b_{2005}, c_{2005}, if $a_0 = 26, b_0 = 6, c_0 = 2004$.

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10 For each polynomial $Q(x)$ let $M(Q)$ be the set of non-negative integers x with $0 < Q(x) < 2004$. We consider polynomials $P_n(x)$ of the form

$$P_n(x) = x^n + a_1 \cdot x^{n-1} + \ldots + a_{n-1} \cdot x + 1$$

with coefficients $a_i \in \{\pm 1\}$ for $i = 1, 2, \ldots, n - 1$.

For each $n = 3^k, k > 0$ determine:

a.) m_n which represents the maximum of elements in $M(P_n)$ for all such polynomials $P_n(x)$

b.) all polynomials $P_n(x)$ for which $|M(P_n)| = m_n$.

– Source: http://www.artofproblemsolving.com/community/c6h49814p315884