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by randomusername

– Individual Competition, Day 1, June 30th 2005

1 For a convex n-gon P_n, we say that a convex quadrangle Q is a *diagonal-quadrangle* of P_n, if its vertices are vertices of P_n and its sides are diagonals of P_n. Let d_n be the number of diagonal-quadrangles of a convex n-gon. Determine d_n for all $n \geq 8$.

2 Determine all polynomials P with integer coefficients satisfying

$$P(P(P(P(P(x))))) = x^{28} \cdot P(P(x)) \qquad \forall x \in \mathbb{R}$$

3 Let $a_0, a_1, a_2, ..., a_n$ be real numbers, which fulfill the following two conditions:
a) $0 = a_0 \leq a_1 \leq a_2 \leq ... \leq a_n$.
b) For all $0 \leq i < j \leq n$ holds: $a_j - a_i \leq j - i$.

Prove that

$$\left(\sum_{i=0}^{n} a_i\right)^2 \geq \sum_{i=0}^{n} a_i^3.$$

– Individual Competition, Day 2, July 1st 2005

4 Determine the smallest natural number $a \geq 2$ for which there exists a prime number p and a natural number $b \geq 2$ such that

$$\frac{a^p - a}{p} = b^2.$$

5 Given is a convex quadrilateral $ABCD$ with $AB = CD$. Draw the triangles ABE and CDF outside $ABCD$ so that $\angle ABE = \angle DCF$ and $\angle BAE = \angle FDC$. Prove that the midpoints of $\overline{AD}$, $\overline{BC}$ and $\overline{EF}$ are collinear.

6 Determine all monotone functions $f : \mathbb{Z} \to \mathbb{Z}$, so that for all $x, y \in \mathbb{Z}$

$$f(x^{2005} + y^{2005}) = (f(x))^{2005} + (f(y))^{2005}$$

– Team Competition, July 2nd 2005

7 For each natural number $n \geq 2$, solve the following system of equations in the integers $x_1, x_2, ..., x_n$:

$$(n^2 - n)x_i + \left(\prod_{j \neq i} x_j\right) S = n^3 - n^2, \qquad \forall 1 \leq i \leq n$$

where

$$S = x_1^2 + x_2^2 + \cdots + x_n^2.$$

8 Given the sets $R_{mn} = \{(x,y) \mid x = 0,1,\ldots,m; y = 0,1,\ldots,n\}$, consider functions $f : R_{mn} \to \{-1,0,1\}$ with the following property: for each quadruple of points $A_1, A_2, A_3, A_4 \in R_{mn}$ which form a square with side length $0 < s < 3$, we have

$$f(A_1) + f(A_2) + f(A_3) + f(A_4) = 0.$$

For each pair (m,n) of positive integers, determine $F(m,n)$, the number of such functions f on R_{mn}.

9 Consider the equation $x^3 + y^3 + z^3 = 2$.
a) Prove that it has infinitely many integer solutions x, y, z.
b) Determine all integer solutions x, y, z with $|x|, |y|, |z| \leq 28$.

10 Determine all pairs (k,n) of non-negative integers such that the following inequality holds $\forall x, y > 0$:

$$1 + \frac{y^n}{x^k} \geq \frac{(1+y)^n}{(1+x)^k}.$$

– Source: http://www.artofproblemsolving.com/community/c6h49814p319242.