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1 Determine all functions $f:(0;\infty)\to\mathbb{R}$ that satisfy

$$f(x+y)=f(x^2+y^2)\quad \forall x,y\in(0;\infty)$$

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2 A parallelogram is inscribed into a regular hexagon so that the centers of symmetry of both figures coincide. Prove that the area of the parallelogram does not exceed $2/3$ the area of the hexagon.

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3 Prove that

$$\sqrt[44]{\tan 1^\circ\cdot\tan 2^\circ\cdot\cdots\cdot\tan 44^\circ}<\sqrt{2}-1<\frac{\tan 1^\circ+\tan 2^\circ+\cdots+\tan 44^\circ}{44}.$$

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4 Let $c\neq 1$ be a positive rational number. Show that it is possible to partition $\mathbb{N}$, the set of positive integers, into two disjoint nonempty subsets A,B so that $x/y\neq c$ holds whenever x and y lie both in A or both in B.

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5 We are given 1978 sets of size 40 each. The size of the intersection of any two sets is exactly 1. Prove that all the sets have a common element.

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6 We are given a family of discs in the plane, with pairwise disjoint interiors. Each disc is tangent to at least six other discs of the family. Show that the family is infinite.

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7 Let M be the set of all lattice points in the plane (i.e. points with integer coordinates, in a fixed Cartesian coordinate system). For any point $P=(x,y)\in M$ we call the points $(x-1,y)$, $(x+1,y)$, $(x,y-1)$, $(x,y+1)$ neighbors of P. Let S be a finite subset of M. A one-to-one mapping f of S onto S is called perfect if $f(P)$ is a neighbor of P, for any $P\in S$. Prove that if such a mapping exists, then there exists also a perfect mapping $g:S\to S$ with the additional property $g(g(P))=P$ for $P\in S$.

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8 For any positive integer k consider the sequence

$$a_n=\sqrt{k+\sqrt{k+\cdots+\sqrt{k}}},$$

where there are n square-root signs on the right-hand side.

(a) Show that the sequence converges, for every fixed integer $k \geq 1$.
(b) Find k such that the limit is an integer. Furthermore, prove that if k is odd, then the limit is irrational.

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9 In a convex polygon P some diagonals have been drawn, without intersections inside P. Show that there exist at least two vertices of P, neither one of them being an endpoint of any one of those diagonals.
