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Individual -

1 The distinct points $X_1, X_2, X_3, X_4, X_5, X_6$ all lie on the same side of the line AB. The six triangles ABX_i are all similar. Show that the X_i lie on a circle.

2 Find all solutions in positive integers to $a^A = b^B = c^C = 1990^{1990}abc$, where $A = b^c, B = c^a, C = a^b$.

3 Show that there are two real solutions to: $x + y^2 + z^4 = 0$ $y + z^2 + x^4 = 0$ $z + x^2 + y^5 = 0$.

4 Find all solutions in positive integers to:

$$\begin{cases} x_1^4 + 14x_1x_2 + 1 = y_1^4 \\ x_2^4 + 14x_2x_3 + 1 = y_2^4 \\ \dots \\ x_n^4 + 14x_nx_1 + 1 = y_n^4 \end{cases}$$

5 Let $n > 1$ be an integer and let f_1, f_2, ..., $f_{n!}$ be the $n!$ permutations of 1, 2, ..., n. (Each f_i is a bijective function from $\{1, 2, ..., n\}$ to itself.) For each permutation f_i, let us define $S(f_i) = \sum_{k=1}^{n} |f_i(k) - k|$. Find $\frac{1}{n!} \sum_{i=1}^{n!} S(f_i)$.

6 $p(x)$ is a polynomial with integer coefficients. The sequence of integers $a_1, a_2, ..., a_n$ (where $n > 2$) satisfies $a_2 = p(a_1), a_3 = p(a_2), ..., a_n = p(a_{n-1}), a_1 = p(a_n)$. Show that $a_1 = a_3$.

Team -

7 D_n is a set of domino pieces. For each pair of non-negative integers (a, b) with $a \le b \le n$, there is one domino, denoted $[a, b]$ or $[b, a]$ in D_n. A *ring* is a sequence of dominoes $[a_1, b_1], [a_2, b_2], ..., [a_k, b_k]$ such that $b_1 = a_2, b_2 = a_3, ..., b_{k-1} = a_k$ and $b_k = a_1$. Show that if n is even there is a ring which uses all the pieces. Show that for n odd, at least $(n + 1)/2$ pieces are not used in any ring. For n odd, how many different sets of $(n + 1)/2$ are there, such that the pieces not in the set can form a ring?

8 We are given a supply of $a \times b$ tiles with a and b distinct positive integers. The tiles are to be used to tile a 28×48 rectangle. Find a, b such that the tile has the smallest possible area and there is only one possible tiling. (If there are two distinct tilings, one of which is a reflection of the other, then we treat that as more than one possible tiling. Similarly for other symmetries.) Find a, b such that the tile has the largest possible area and there is more than one possible tiling.

9 $a_1, a_2, ..., a_n$ is a sequence of integers such that every non-empty subsequence has non-zero sum. Show that we can partition the positive integers into a finite number of sets such that if x_i all belong to the same set, then $a_1x_1 + a_2x_2 + ... + a_nx_n$ is non-zero.