VMO 2020

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–	Day 1
1	Let a sequence (x_n) satisfy :$x_1 = 1$ and $x_{n+1} = x_n + 3\sqrt{x_n} + \frac{n}{\sqrt{x_n}}$,∀n$\geq 1$ a) Prove lim$\frac{n}{x_n} = 0$ b) Find lim$\frac{n^2}{x_n}$
2	a)Let$a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$.Prove that: $\|a - b\| + \|b - c\| + \|c - a\| \leq 2\sqrt{2}$ b) Let $a_1, a_2, ..a_{2019} \in \mathbb{R}$ and $\sum_{i=1}^{2019} a_i^2 = 1$.Find the maximum of: $S = \|a_1 - a_2\| + \|a_2 - a_3\| + ... + \|a_{2019} - a_1\|$
3	Let a sequence (a_n) satisfy: $a_1 = 5, a_2 = 13$ and $a_{n+1} = 5a_n - 6a_{n-1}, \forall n \geq 2$ a) Prove that $(a_n, a_{n+1}) = 1, \forall n \geq 1$ b) Prove that: $2^{k+1}\|p - 1 \forall k \in \mathbb{N}$, if p is a prime factor of a_{2^k}
4	Let a non-isosceles acute triangle ABC with the circumscribed cycle (O) and the orthocenter H. D, E, F are the reflection of O in the lines BC, CA and AB. a) H_a is the reflection of H in BC, A' is the reflection of A at O and O_a is the center of (BOC). Prove that H_aD and OA' intersect on (O). b) Let X is a point satisfy AXDA' is a parallelogram. Prove that (AHX), (ABF), (ACE) have a comom point different than A
–	Day 2
5	Let a system of equations: $\begin{cases} x - ay = yz \\ y - az = zx \\ z - ax = xy \end{cases}$ a)Find (x,y,z) if a=0 b)Prove that: the system have 5 distinct roots ∀a¿1,a$\in \mathbb{R}$.
6	Let a non-isosceles acute triangle ABC with tha attitude AD, BE, CF and the orthocenter H. DE, DF intersect (AD) at M, N respectively. $P \in AB, Q \in AC$ satisfy $NP \perp AB, MQ \perp AC$ a) Prove that EF is the tangent line of (APQ) b) Let T be the tangency point of (APQ) with EF,.DT ∩ MN=K. L is the reflection of A in MN. Prove that MN, EF ,(DLK) pass through a piont
7	Given a positive integer $n > 1$. Denote T a set that contains all ordered sets $(x; y; z)$ such that

x, y, z are all distinct positive integers and $1 \leq x, y, z \leq 2n$. Also, a set A containing ordered sets $(u; v)$ is called *"connected"* with T if for every $(x; y; z) \in T$ then $\{(x; y), (x; z), (y; z)\} \cap A \neq \varnothing$.
a) Find the number of elements of set T.
b) Prove that there exists a set "connected" with T that has exactly $2n(n-1)$ elements.
c) Prove that every set "connected" with T has at least $2n(n-1)$ elements.