**7th Gulf Mathematical Olympiad 2019**

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**1** Let $ABCD$ be a trapezium (trapezoid) with $AD$ parallel to $BC$ and $J$ be the intersection of the diagonals $AC$ and $BD$. Point $P$ a chosen on the side $BC$ such that the distance from $C$ to the line $AP$ is equal to the distance from $B$ to the line $DP$.

*The following three questions 1, 2 and 3 are independent, so that a condition in one question does not apply in another question.*

1.Suppose that $Area(\triangle\ AJB) = 6$ and that $Area(\triangle\ BJC) = 9$. Determine $Area(\triangle\ APD)$.
2. Find all points $Q$ on the plane of the trapezium such that $Area(\triangle\ AQB) = Area(\triangle\ DQC)$.
3. Prove that $PJ$ is the angle bisector of $\angle APD$.

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**2** 1. Find $N$, the smallest positive multiple of $45$ such that all of its digits are either $7$ or $0$.
2. Find $M$, the smallest positive multiple of $32$ such that all of its digits are either $6$ or $1$.
3. How many elements of the set $\{1, 2, 3, ..., 1441\}$ have a positive multiple such that all of its digits are either $5$ or $2$?

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**3** Consider the set $S = \{1, 2, 3, ..., 1441\}$.
1. Nora counts thoses subsets of $S$ having exactly two elements, tbe sum of which is even. Rania counts those subsets of $S$ having exactly two elements, the sum of which is odd. Determine the numbers counted by Nora and Rania.
2. Let $t$ be the number of subsets of $S$ which have at least two elements and the product of the elements is even. Determine the greatest power of $2$ which divides $t$.
3. Ahmad counts the subsets of $S$ having $77$ elements such that in each subset the sum of the elements is even. Bushra counts the subsets of $S$ having $77$ elements such that in each subset the sum of the elements is odd. Whose number is bigger? Determine the difference between the numbers found by Ahmad and Bushra.

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**4** Consider the sequence $(a_n)_{n\ge 1}$ defined by $a_n = n$ for $n \in \{1, 2, 3.4, 5, 6\}$, and for $n \ge 7$:

$$a_n = \lfloor \frac{a_1 + a_2 + ... + a_{n-1}}{2} \rfloor$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. For example : $\lfloor 2.4 \rfloor = 2$, $\lfloor 3 \rfloor = 3$ and $\lfloor \pi \rfloor = 3$.

For all integers $n \ge 2$, let $S_n = \{a_1, a_1, ..., a_n\} - \{r_n\}$ where $r_n$ is the remainder when $a_1 + a_2 + ... + a_n$ is divided by $3$. The minus $-$ denotes the "*remove it if it is there*" notation. For example : $S_4 = 2, 3, 4$ because $r_4 = 1$ so $1$ is removed from $\{1, 2, 3, 4\}$. However $S_5 = \{1, 2, 3, 4, 5\}$ betawe $r_5 = 0$ and $0$ is not in the set $\{1, 2, 3, 4, 5\}$.
1. Determine $S_7, S_8, S_9$ and $S_{10}$.

2. We say that a set $S_n$ for $n \geq 6$ is well-balanced if it can be partitioned into three pairwise disjoint subsets with equal sum. For example : $S_6 = \{1,2,3,4,5,6\} = \{1,6\} \cup \{2,5\} \cup \{3,4\}$ and $1+6=2+5=3+4$. Prove that $S_7, S_8, S_9$ and $S_{10}$ are well-balanced .
3. Is the set $S_{2019}$ well-balanced? Justify your answer.