European Mathematical Cup 2019

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– Junior Division

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1 Every positive integer is marked with a number from the set $\{0,1,2\}$, according to the following rule:

if a positive integer k is marked with $j,$ then the integer $k+j$ is marked with 0.

Let S denote the sum of marks of the first 2019 positive integers. Determine the maximum possible value of S.

Proposed by Ivan Novak

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2 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence defined recursively such that $x_1=\sqrt{2}$ and

$$x_{n+1}=x_n+\frac{1}{x_n} \text{ for } n\in\mathbb{N}.$$

Prove that the following inequality holds:

$$\frac{x_1^2}{2x_1x_2-1}+\frac{x_2^2}{2x_2x_3-1}+\dots+\frac{x_{2018}^2}{2x_{2018}x_{2019}-1}+\frac{x_{2019}^2}{2x_{2019}x_{2020}-1}>\frac{2019^2}{x_{2019}^2+\frac{1}{x_{2019}^2}}.$$

Proposed by Ivan Novak

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3 Let ABC be a triangle with circumcircle ω. Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B \parallel l_C$. The second intersections of l_B and l_C with ω are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B \parallel OP \parallel l_C$.

Proposed by Stefan Lozanovski, Macedonia

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4 Let u be a positive rational number and m be a positive integer. Define a sequence $q_1,q_2,q_3,\dots$ such that $q_1=u$ and for $n\geqslant 2$:

$$\text{if } q_{n-1}=\frac{a}{b} \text{ for some relatively prime positive integers } a \text{ and } b, \text{ then } q_n=\frac{a+mb}{b+1}.$$

Determine all positive integers m such that the sequence $q_1,q_2,q_3,\dots$ is eventually periodic for any positive rational number u.

Remark: A sequence $x_1, x_2, x_3, \ldots$ is *eventually periodic* if there are positive integers c and t such that $x_n = x_{n+t}$ for all $n \geqslant c$.

Proposed by Petar Nizié-Nikolac

– Senior Division

1 For positive integers a and b, let $M(a,b)$ denote their greatest common divisor. Determine all pairs of positive integers (m,n) such that for any two positive integers x and y such that $x \mid m$ and $y \mid n$,

$$M(x+y, mn) > 1.$$

Proposed by Ivan Novak

2 Let n be a positive integer. An $n \times n$ board consisting of n^2 cells, each being a unit square colored either black or white, is called *convex* if for every black colored cell, both the cell directly to the left of it and the cell directly above it are also colored black. We define the *beauty* of a board as the number of pairs of its cells (u,v) such that u is black, v is white, and u and v are in the same row or column. Determine the maximum possible beauty of a convex $n \times n$ board.

Proposed by Ivan Novak

3 In an acute triangle ABC with $|AB| \neq |AC|$, let I be the incenter and O the circumcenter. The incircle is tangent to $\overline{BC}, \overline{CA}$ and $\overline{AB}$ in D, E and F respectively. Prove that if the line parallel to EF passing through I, the line parallel to AO passing through D and the altitude from A are concurrent, then the point of concurrence is the orthocenter of the triangle ABC.

Proposed by Petar Nizié-Nikolac

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

for all $x, y \in \mathbb{R}$.

Proposed by Adrian Beker