**Rioplatense Mathematical Olympiad, Level 3 2019**

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– Day 1

**1** Let $ABCDEF$ be a regular hexagon, in the sides $AB$, $CD$, $DE$ and $FA$ we choose four points $P, Q, R$ and $S$ respectively, such that $PQRS$ is a square. Prove that $PQ$ and $BC$ are parallel.

**2** Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(f(x)^2 + f(y^2)) = (x-y)f(x-f(y))$

**3** In the dog dictionary the words are any sequence of letters $A$ and $U$ for example $AA$, $UAU$ and $AUAU$. For each word, your "profundity" will be the quantity of subwords we can obtain by the removal of some letters.
For each positive integer $n$, determine the largest "profundity" of word, in dog dictionary, can have with $n$ letters.
Note: The word $AAUUA$ has "profundity" $14$ because your subwords are $A, U, AU, AA, UU, UA, AUU, UUA$

– Day 2

**4** Prove that there are infinite triples $(a, b, c)$ of positive integers $a, b, c > 1$, $gcd(a, b) = gcd(b, c) = gcd(c, a) = 1$ such that $a + b + c$ divides $a^b + b^c + c^a$.

**5** Let $ABC$ be a triangle with $AB < AC$ and circuncircle $\omega$. Let $M$ and $N$ be the midpoints of $AC$ and $AB$ respectively and $G$ is the centroid of $ABC$. Let $P$ be the foot of perpendicular of $A$ to the line $BC$, and the point $Q$ is the intersection of $GP$ and $\omega$($Q, P, G$ are collinears in this order). The line $QM$ cuts $\omega$ in $M_1$ and the line $QN$ cuts $\omega$ in $N_1$. If $K$ is the intersection of $BM_1$ and $CN_1$ prove that $P$, $G$ and $K$ are collinears.

**6** Let $\alpha > 1$ be a real number such that the sequence $a_n = \alpha\lfloor\alpha^n\rfloor - \lfloor\alpha^{n+1}\rfloor$, with $n \geq 1$, is periodic, that is, there is a positive integer $p$ such that $a_{n+p} = a_n$ for all $n$. Prove that $\alpha$ is an integer.