Mathematical Olympiad 2019

www.artofproblemsolving.com/community/c1041013

by parmenides51

1 Find all functions $f : R \to R$ such that $f(2xy) + f(f(x + y)) = xf(y) + yf(x) + f(x + y)$ for all real numbers x and y.

---

2 Twelve students participated in a theater festival consisting of n different performances. Suppose there were six students in each performance, and each pair of performances had at most two students in common. Determine the largest possible value of n.

---

3 Find all triples (a, b, c) of positive integers such that $a^2 + b^2 = n \cdot lcm(a, b) + n^2$ $b^2 + c^2 = n \cdot lcm(b, c) + n^2$ $c^2 + a^2 = n \cdot lcm(c, a) + n^2$
for some positive integer n.

---

4 In acute triangle $ABC$with $\angle BAC > \angle BCA$, let P be the point on side BC such that $\angle PAB = \angle BCA$. The circumcircle of triangle APB meets side AC again at Q. Point D lies on segment AP such that $\angle QDC = \angle CAP$.
Point E lies on line BD such that $CE = CD$. The circumcircle of triangle CQE meets segment CD again at F, and line QF meets side BC at G. Show that B, D, F, and G are concyclic

---