Dutch Mathematical Olympiad 2019

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1 A *complete* number is a 9 digit number that contains each of the digits 1 to 9 exactly once. The *difference* number of a number N is the number you get by taking the differences of consecutive digits in N and then stringing these digits together. For instance, the *difference* number of 25143 is equal to 3431. The *complete* number 124356879 has the additional property that its *difference* number, 12121212, consists of digits alternating between 1 and 2.
Determine all a with $3 \le a \le 9$ for which there exists a *complete* number N with the additional property that the digits of its *difference* number alternate between 1 and a.

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2 There are n guests at a party. Any two guests are either friends or not friends. Every guest is friends with exactly four of the other guests. Whenever a guest is not friends with two other guests, those two other guests cannot be friends with each other either.
What are the possible values of n?

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3 Points A, B, and C lie on a circle with centre M. The reflection of point M in the line AB lies inside triangle ABC and is the intersection of the angle bisectors of angles A and B. Line AM intersects the circle again in point D.
Show that $|CA| \cdot |CD| = |AB| \cdot |AM|$.

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4 The sequence of Fibonacci numbers $F_0, F_1, F_2, ...$ is defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for all $n > 0$. For example, we have $F_2 = F_0 + F_1 = 2, F_3 = F_1 + F_2 = 3, F_4 = F_2 + F_3 = 5$, and $F_5 = F_3 + F_4 = 8$. The sequence $a_0, a_1, a_2, ...$ is defined by $a_n = \frac{1}{F_n F_{n+2}}$ for all $n \ge 0$.
Prove that for all $m \ge 0$ we have: $a_0 + a_1 + a_2 + ... + a_m < 1$.

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5 Thomas and Nils are playing a game. They have a number of cards, numbered $1, 2, 3$, et cetera. At the start, all cards are lying face up on the table. They take alternate turns. The person whose turn it is, chooses a card that is still lying on the table and decides to either keep the card himself or to give it to the other player. When all cards are gone, each of them calculates the sum of the numbers on his own cards. If the difference between these two outcomes is divisible by 3, then Thomas wins. If not, then Nils wins.
(a) Suppose they are playing with 2018 cards (numbered from 1 to 2018) and that Thomas starts. Prove that Nils can play in such a way that he will win the game with certainty.
(b) Suppose they are playing with 2020cards (numbered from 1 to 2020) and that Nils starts. Which of the two players can play in such a way that he wins with certainty?
