Dutch IMO TST Team Selection Test 2012

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–	Day 1
1	A line, which passes through the incentre I of the triangle ABC, meets its sides AB and BC at the points M and N respectively. The triangle BMN is acute. The points K, L are chosen on the side AC such that $\angle ILA = \angle IMB$ and $\angle KC = \angle INB$. Prove that $AM + KL + CN = AC$. *S. Berlov*
2	Let a, b, c and d be positive real numbers. Prove that $$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}\geq 0$$
3	Determine all positive integers that cannot be written as $\frac{a}{b}+\frac{a+1}{b+1}$ where a and b are positive integers.
4	Let n be a positive integer divisible by 4. We consider the permutations $(a_1, a_2, ..., a_n)$ of $(1, 2, ..., n)$ having the following property: for each j we have $a_i + j = n + 1$ where $i = a_j$. Prove that there are exactly $\frac{(\frac{1}{2}n)!}{(\frac{1}{4}n)!}$ such permutations.
5	Let Γ be the circumcircle of the acute triangle ABC. The angle bisector of angle ABC intersects AC in the point B_1 and the short arc AC of Γ in the point P. The line through B_1 perpendicular to BC intersects the short arc BC of Γ in K. The line through B perpendicular to AK intersects AC in L. Prove that K, L and P lie on a line.
–	Day 2
1	For all positive integers a and b, we de ne $a@b = \frac{a-b}{gcd(a,b)}$. Show that for every integer $n > 1$, the following holds: n is a prime power if and only if for all positive integers m such that $m < n$, it holds that $gcd(n, n@m) = 1$.
2	There are two boxes containing balls. One of them contains m balls, and the other contains n balls, where $m, n > 0$. Two actions are permitted: (i) Remove an equal number of balls from both boxes. (ii) Increase the number of balls in one of the boxes by a factor k. Is it possible to remove all of the balls from both boxes with just these two actions,

1. if $k = 2$?
2. if $k = 3$?

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3 Determine all pairs (x, y) of positive integers satisfying $x + y + 1|2xy$ and $x + y - 1|x^2 + y^2 - 1$.

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4 Let $\triangle\ ABC$ be a triangle. The angle bisector of $\angle CAB$ intersectsBC at L. On the interior of line segments AC and AB, two points, M and N, respectively, are chosen in such a way that the lines AL, BM and CN are concurrent, and such that $\angle AMN = \angle ALB$. Prove that $\angle NML = 90^o$.

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5 Find all functions $f : R \to R$ satisfying $f(x + xy + f(y)) = (f(x) + \frac{1}{2})(f(y) + \frac{1}{2})$ for all $x, y \in R$.
