Dutch BxMO Team Selection Test 2019

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1 Prove that for each positive integer n there are at most two pairs (a, b) of positive integers with following two properties:
(i) $a^2 + b = n$,
(ii) $a + b$ is a power of two, i.e. there is an integer $k \geq 0$ such that $a + b = 2^k$.

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2 Let ΔABC be a triangle with an inscribed circle centered at I. The line perpendicular to AI at I intersects $\odot(ABC)$ at P, Q such that, P lies closer to B than C. Let $\odot(BIP) \cap \odot(CIQ) = S$. Prove that, SI is the angle bisector of $\angle PSQ$

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3 Let x and y be positive real numbers.
1. Prove: if $x^3 - y^3 \geq 4x$, then $x^2 > 2y$.
2. Prove: if $x^5 - y^3 \geq 2x$, then $x^3 \geq 2y$.

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4 Do there exist a positive integer k and a non-constant sequence $a_1, a_2, a_3, ...$ of positive integers such that $a_n = gcd(a_{n+k}, a_{n+k+1})$ for all positive integers n?

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5 In a country, there are 2018 cities, some of which are connected by roads. Each city is connected to at least three other cities. It is possible to travel from any city to any other city using one or more roads. For each pair of cities, consider the shortest route between these two cities. What is the greatest number of roads that can be on such a shortest route?
