Dutch IMO Team Selection Test 2019

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– Day 1

1 Let $P(x)$ be a quadratic polynomial with two distinct real roots.
For all real numbers a and b satisfying $|a|, |b| \geq 2017$, we have $P(a^2 + b^2) \geq P(2ab)$.
Show that at least one of the roots of P is negative.

2 Write S_n for the set $\{1, 2, ..., n\}$. Determine all positive integers n for which there exist functions $f : S_n \to S_n$ and $g : S_n \to S_n$ such that for every x exactly one of the equalities $f(g(x)) = x$ and $g(f(x)) = x$ holds.

3 Let n be a positive integer. Determine the maximum value of $gcd(a, b) + gcd(b, c) + gcd(c, a)$ for positive integers a, b, c such that $a + b + c = 5n$.

4 Let ΔABC be a scalene triangle. Points D, E lie on side $\overline{AC}$ in the order, A, E, D, C. Let the parallel through E to BC intersect $\odot(ABD)$ at F, such that, E and F lie on the same side of AB. Let the parallel through E to AB intersect $\odot(BDC)$ at G, such that, E and G lie on the same side of BC. Prove, Points D, F, E, G are concyclic

– Day 2

1 In each of the different grades of a high school there are an odd number of pupils. Each pupil has a best friend (who possibly is in a different grade). Everyone is the best friend of their best friend. In the upcoming school trip, every pupil goes to either Rome or Paris. Show that the pupils can be distributed over the two destinations in such a way that
(i) every student goes to the same destination as their best friend;
(ii) for each grade the absolute difference between the number of pupils that are going to Rome and that of those who are going to Paris is equal to 1.

2 Determine all 4-tuples (a, b, c, d) of positive real numbers satisfying $a + b + c + d = 1$ and $\max(\frac{a^2}{b}, \frac{b^2}{a}) \cdot \max(\frac{c^2}{d}, \frac{d^2}{c}) = (\min(a + b, c + d))^4$

3 Let ABC be an acute angles triangle with O the center of the circumscribed circle. Point Q lies on the circumscribed circle of $\triangle BOC$ so that OQ is a diameter. Point M lies on CQ and point N lies internally on line segment BC so that $ANCM$ is a parallelogram. Prove that the circumscribed circle of $\triangle BOC$ and the lines AQ and NM pass through the same point.

4 Find all functions $f : Z \to Z$ satisfying • $f(p) > 0$ for all prime numbers p, • $p|(f(x)+f(p))^{f(p)} - x$ for all $x \in Z$ and all prime numbers p.

– Day 3

1 Let $ABCD$ be a cyclic quadrilateral (In the same order) inscribed into the circle $\odot(O)$. Let $\overline{AC} \cap \overline{BD} = E$. A randome line ℓ through E intersects $\overline{AB}$ at P and BC at Q. A circle ω touches ℓ at E and passes through D. Given, $\omega \cap \odot(O) = R$. Prove, Points B, Q, R, P are concyclic.

2 Let n be a positive integer. Prove that n^2+n+1 cannot be written as the product of two positive integers of which the difference is smaller than $2\sqrt{n}$.

3 Find all functions $f : Z \to Z$ satisfying the following two conditions:
(i) for all integers x we have $f(f(x)) = x$,
(ii) for all integers x and y such that $x + y$ is odd, we have $f(x) + f(y) \geq x + y$.

4 There are 300 participants to a mathematics competition. After the competition some of the contestants play some games of chess. Each two contestants play at most one game against each other. There are no three contestants, such that each of them plays against each other. Determine the maximum value of n for which it is possible to satisfy the following conditions at the same time: each contestant plays at most n games of chess, and for each m with $1 \leq m \leq n$, there is a contestant playing exactly m games of chess.