Federal Competition For Advanced Students, Part 2 2014

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by parmenides51

–	Day 1
1	For each positive natural number n let $d(n)$ be the number of its divisors including 1 and n. For which positive natural numbers n, for every divisor t of n, that $d(t)$ is a divisor of $d(n)$?
2	Let S be the set of all real numbers greater than or equal to 1. Determine all functions $f: S \to S$, so that for all real numbers $x, y \in S$ with $x^2 - y^2 \in S$ the condition $f(x^2 - y^2) = f(xy)$ is fulfilled.
3	(i) For which triangles with side lengths a, b and c apply besides the triangle inequalities $a+b > c, b+c > a$ and $c+a > b$ also the inequalities $a^2+b^2 > c^2, b^2+c^2 > a^2$ and $a^2+c^2 > b^2$? (ii) For which triangles with side lengths a, b and c apply besides the triangle inequalities $a+b > c, b+c > a$ and $c+a > b$ also for all positive natural n the inequalities $a^n+b^n > c^n, b^n+c^n > a^n$ and $a^n+c^n > b^n$?
–	Day 2
4	For an integer n let $M(n) = \{n, n+1, n+2, n+3, n+4\}$. Furthermore, be $S(n)$ sum of squares and $P(n)$ the product of the squares of the elements of $M(n)$. For which integers n is $S(n)$ a divisor of $P(n)$?
5	Show that the inequality $(x^2+y^2z^2)(y^2+x^2z^2)(z^2+x^2y^2) \geq 8xy^2z^3$ is valid for all integers x, y and z.When does equality apply?
6	Let U be the center of the circumcircle of the acute-angled triangle ABC. Let M_A, M_B and M_C be the circumcenters of triangles UBC, UAC and UAB respecrively. For which triangles ABC is the triangle $M_AM_BM_C$ similar to the starting triangle (with a suitable order of the vertices)?