Czech And Slovak Mathematical Olympiad, Round III, Category A 2005

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by parmenides51

1 Consider all arithmetical sequences of real numbers $(x_i)^\infty = 1$ and $(y_i)^\infty = 1$ with the common first term, such that for some $k > 1, x_{k-1}y_{k-1} = 42, x_k y_k = 30$, and $x_{k+1}y_{k+1} = 16$. Find all such pairs of sequences with the maximum possible k.

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2 Determine for which m there exist exactly 2^{15} subsets X of $\{1, 2, ..., 47\}$ with the following property: m is the smallest element of X, and for every $x \in X$, either $x + m \in X$ or $x + m > 47$.

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3 In a trapezoid $ABCD$ with $AB//CD, E$ is the midpoint of BC. Prove that if the quadrilaterals $ABED$ and $AECD$ are tangent, then the sides $a = AB, b = BC, c = CD, d = DA$ of the trapezoid satisfy the equalities $a + c = \frac{b}{3} + d$ and $\frac{1}{a} + \frac{1}{c} = \frac{3}{b}$.

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4 An acute-angled triangle AKL is given on a plane. Consider all rectangles $ABCD$ circumscribed to triangle AKL such that point K lies on side BC and point L lieson side CD. Find the locus of the intersection S of the diagonals AC and BD.

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5 Let p, q, r, s be real numbers with $q \neq -1$ and $s \neq -1$. Prove that the quadratic equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root, while their other roots are inverse of each other, if and only if $pr = (q + 1)(s + 1)$ and $p(q + 1)s = r(s + 1)q$.
(A double root is counted twice.)

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6 Decide whether for every arrangement of the numbers $1, 2, 3, ..., 15$ in a sequence one can color these numbers with at most four different colors in such a way that the numbers of each color form a monotone subsequence.
