The problems from the CCA Math Bonanza held on 1/18/2020

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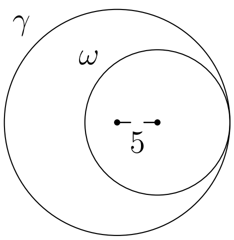
by mira74

– Individual Round

I1 An ant is crawling along the coordinate plane. Each move, it moves one unit up, down, left, or right with equal probability. If it starts at $(0,0)$, what is the probability that it will be at either $(2,1)$ or $(1,2)$ after 6 moves?

2020 CCA Math Bonanza Individual Round#1

I2 Circles ω and γ are drawn such that ω is internally tangent to γ, the distance between their centers are 5, and the area inside of γ but outside of ω is 100π. What is the sum of the radii of the circles?



2020 CCA Math Bonanza Individual Round#2

I3 Compute the remainder when $\left(\frac{2^5}{2}\right)^5$ is divided by 5.

2020 CCA Math Bonanza Individual Round#3

I4 Alan, Jason, and Shervin are playing a game with MafsCounts questions. They each start with 2 tokens. In each round, they are given the same MafsCounts question. The first person to solve the MafsCounts question wins the round and steals one token from each of the other players in the game. They all have the same probability of winning any given round. If a player runs out of tokens, they are removed from the game. The last player remaining wins the game.

If Alan wins the first round but does not win the second round, what is the probability that he wins the game?

2020 CCA Math Bonanza Individual Round#4

I5 Let $f(x) = x^2 - kx + (k-1)^2$ for some constant k. What is the largest possible real value of k such that f has at least one real root?

2020 CCA Math Bonanza Individual Round#5

I6 Let P be a point outside a circle Γ centered at point O, and let PA and PB be tangent lines to circle Γ. Let segment PO intersect circle Γ at C. A tangent to circle Γ through C intersects PA and PB at points E and F, respectively. Given that $EF = 8$ and $\angle APB = 60^\circ$, compute the area of $\triangle AOC$.

2020 CCA Math Bonanza Individual Round#6

I7 Define the binary operation $a\Delta b = ab + a - 1$. Compute

$$10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta(10\Delta10))))))))$$

where 10 is written 10 times.

2020 CCA Math Bonanza Individual Round#7

I8 Compute the remainder when the largest integer below $\frac{3^{123}}{5}$ is divided by 16.

2020 CCA Math Bonanza Individual Round#8

I9 A sequence a_n of real numbers satisfies $a_1 = 1$, $a_2 = 0$, and $a_n = (S_{n-1}+1)S_{n-2}$ for all integers $n \geq 3$, where $S_k = a_1 + a_2 + \cdots + a_k$ for positive integers k. What is the smallest integer $m > 2$ such that 127 divides a_m?

2020 CCA Math Bonanza Individual Round#9

I10 Annie takes a 6 question test, with each question having two parts each worth 1 point. On each **part**, she receives one of nine letter grades $\{$A,B,C,D,E,F,G,H,I$\}$ that correspond to a unique numerical score. For each **question**, she receives the sum of her numerical scores on both parts. She knows that A corresponds to 1, E corresponds to 0.5, and I corresponds to 0.

When she receives her test, she realizes that she got two of each of A, E, and I, and she is able to determine the numerical score corresponding to all 9 markings. If n is the number of ways she can receive letter grades, what is the exponent of 2 in the prime factorization of n?

2020 CCA Math Bonanza Individual Round#10

I11 Points C, A, D, M, E, B, F lie on a line in that order such that $CA = AD = EB = BF = 1$ and M is the midpoint of DB. Let X be a point such that a quarter circle arc exists with center D and endpoints C, X. Suppose that line XM is tangent to the unit circle centered at B. Compute AB.

2020 CCA Math Bonanza Individual Round#11

I12 Find all pairs (a,b) of positive integers satisfying the following conditions:
- $a \leq b$
- ab is a perfect cube
- No divisor of a or b is a perfect cube greater than 1
- $a^2+b^2=85\text{lcm}(a,b)$

2020 CCA Math Bonanza Individual Round#12

I13 Let n be a positive integer. Compute, in terms of n, the number of sequences $(x_1,\ldots,x_{2n})$ with each $x_i\in\{0,1,2,3,4\}$ such that $x_1^2+\dots+x_{2n}^2$ is divisible by 5.

2020 CCA Math Bonanza Individual Round#13

I14 An ant starts at the point $(0,0)$ in the coordinate plane. It can make moves from lattice point (x_1,y_1) to lattice point (x_2,y_2) whenever $x_2\geq x_1$, $y_2\geq y_1$, and $(x_1,y_1)\neq(x_2,y_2)$. For all non-negative integers m,n, define $a_{m,n}$ to be the number of possible sequences of moves from $(0,0)$ to (m,n) (e.g. $a_{0,0}=1$ and $a_{1,1}=3$). Compute

$$\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{a_{m,n}}{10^{m+n}}.$$

2020 CCA Math Bonanza Individual Round#14

I15 Let θ be an obtuse angle with $\sin\theta=\frac{3}{5}$. If an ant starts at the origin and repeatedly moves 1 unit and turns by an angle of θ, there exists a region R in the plane such that for every point $P\in R$ and every constant $c>0$, the ant is within a distance c of P at some point in time (so the ant gets arbitrarily close to every point in the set). What is the largest possible area of R?

2020 CCA Math Bonanza Individual Round#15

– Team Round

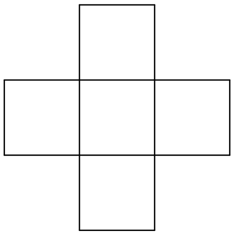
T1 Compute the number of permutations of $\{1,2,3\}$ with the property that there is some number that can be removed such that the remaining numbers are in increasing order. For example, $(2,1,3)$ has this property because removing 1 leaves $(2,3)$, which is in increasing order.

2020 CCA Math Bonanza Team Round#1

T2 The base 4 repeating decimal $0.\overline{12}_4$ can be expressed in the form $\frac{a}{b}$ in base 10, where a and b are relatively prime positive integers. Compute the sum of a and b.

2020 CCA Math Bonanza Team Round#2

T3 Five unit squares are arranged in a plus shape as shown below:



What is the area of the smallest circle containing the interior and boundary of the plus shape?

2020 CCA Math Bonanza Team Round#3

T4 Compute

$$\left(\frac{4-\log_{36} 4-\log_6 18}{\log_4 3}\right)\cdot\left(\log_8 27+\log_2 9\right).$$

2020 CCA Math Bonanza Team Round#4

T5 Find all pairs of real numbers (x, y) satisfying both equations

$$3x^2+3xy+2y^2=2$$
$$x^2+2xy+2y^2=1.$$

2020 CCA Math Bonanza Team Round#5

T6 A cat can see 1 mile in any direction. The cat walks around the 13 mile perimeter of a triangle. Over the course of its walk, it sees every point inside of this triangle. What is the largest possible area, in square miles, of the total region it sees?

2020 CCA Math Bonanza Team Round#6

T7 Compute the remainder when $99989796\ldots121110090807\ldots01$ is divided by $010203\ldots091011\ldots9798$ (note that the first one starts at 99, and the second one ends at 98).

2020 CCA Math Bonanza Team Round#7

T8 Call an *ordered* triple (a, b, c) [i]d-tall[/i] if there exists a triangle with side lengths a, b, c and the height to the side with length a is d. Suppose that for some positive integer k, there are exactly 210 k-tall ordered triples of positive integers. How many k-tall ordered triples (a, b, c) are there such that a triangle ABC with $BC=a, CA=b, AB=c$ satisfies both $\angle B<90^\circ$ and $\angle C<90^\circ$?

2020 CCA Math Bonanza Team Round#8

T9 A game works as follows: the player pays 2 tokens to enter the game. Then, a fair coin is flipped. If the coin lands on heads, they receive 3 tokens; if the coin lands on tails, they receive nothing. A player starts with 2 tokens and keeps playing this game until they do not have enough tokens to play again. What is the expected value of the number of tokens they have left at the end?

2020 CCA Math Bonanza Team Round#9

T10 In $\triangle ABC$ with an obtuse angle at A, let D be the foot of the A altitude and E be the foot of the B altitude. If $AC+CD=DB$ and $BC-AE=EC$, compute $\angle A$ in degrees.

2020 CCA Math Bonanza Team Round#10

– Lightning Round

L1.1 We know that 201 and 9 give the same remainder when divided by 24. What is the smallest positive integer k such that $201+k$ and $9+k$ give the same remainder when divided by 24?

2020 CCA Math Bonanza Lightning Round#1.1

L1.2 Let $a_1=3$, $a_2=7$, and $a_3=1$. Let $b_0=0$ and for all positive integers n, let $b_n=10b_{n-1}+a_n$. Compute $b_1\times b_2\times b_3$.

2020 CCA Math Bonanza Lightning Round#1.2

L1.3 If $ABCDE$ is a regular pentagon and X is a point in its interior such that CDX is equilateral, compute $\angle AXE$ in degrees.

2020 CCA Math Bonanza Lightning Round#1.3

L1.4 Let ABC be a triangle with $AB=3$, $BC=4$, and $CA=5$. Points A_1, B_1, and C_1 are chosen on its incircle. Compute the maximum possible sum of the areas of triangles A_1BC, AB_1C, and ABC_1.

2020 CCA Math Bonanza Lightning Round#1.4

L2.1 We know that 201 and 9 give the same remainder when divided by 24. What is the smallest positive integer k such that $201+k$ and $9+k$ give the same remainder when divided by $24+k$?

2020 CCA Math Bonanza Lightning Round#2.1

L2.2 A rectangular box with side lengths 1, 2, and 16 is cut into two congruent smaller boxes with integer side lengths. Compute the square of the largest possible length of the space diagonal of one of the smaller boxes.

2020 CCA Math Bonanza Lightning Round#2.2

L2.3 3 uncoordinated aliens launch a 3-day attack on 4 galaxies. Each day, each of the three aliens chooses a galaxy uniformly at random from the remaining galaxies and destroys it. They make their choice simultaneously and independently, so two aliens could destroy the same galaxy. If the probability that every galaxy is destroyed by the end of the attack can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, what is $m+n$?

2020 CCA Math Bonanza Lightning Round#2.3

L2.4 If

$$\sum_{k=1}^{1000}\left(\frac{k+1}{k}+\frac{k}{k+1}\right)=\frac{m}{n}$$

for relatively prime positive integers m, n, compute $m+n$.

2020 CCA Math Bonanza Lightning Round#2.4

L3.1 For some positive integer n, the sum of all odd positive integers between n^2-n and n^2+n is a number between 9000 and 10000, inclusive. Compute n.

2020 CCA Math Bonanza Lightning Round#3.1

L3.2 Archit and Ayush are walking around on the set of points (x, y) for all integers $-1 \leq x, y \leq 1$. Archit starts at $(1,1)$ and Ayush starts at $(1,0)$. Each second, they move to another point in the set chosen uniformly at random among the points with distance 1 away from them. If the probability that Archit goes to the point $(0,0)$ strictly before Ayush does can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m+n$.

2020 CCA Math Bonanza Lightning Round#3.2

L3.3 Compute the largest prime factor of $111^2+11^3+1^1$.

2020 CCA Math Bonanza Lightning Round#3.3

L3.4 Willy Wonka has n distinguishable pieces of candy that he wants to split into groups. If the number of ways for him to do this is $p(n)$, then we have

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	5	15	52	203	877	4140	21147	115975

Define a *splitting* of the n distinguishable pieces of candy to be a way of splitting them into groups. If Willy Wonka has 8 candies, what is the sum of the number of groups over all splittings he can use?

2020 CCA Math Bonanza Lightning Round#3.4

L4.1 Alice picks a number uniformly at random from the first 5 even positive integers, and Palice picks a number uniformly at random from the first 5 odd positive integers. If Alice picks a larger number than Palice with probability $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m+n$.

2020 CCA Math Bonanza Lightning Round#4.1

L4.2 Let $a_0, a_1, \ldots$ be a sequence of positive integers such that $a_0 = 1$, and for all positive integers n, a_n is the smallest composite number relatively prime to all of $a_0, a_1, \ldots, a_{n-1}$. Compute a_{10}.

2020 CCA Math Bonanza Lightning Round#4.2

L4.3 Let $ABCD$ be a convex quadrilateral such that $AB = 4$, $BC = 5$, $CA = 6$, and $\triangle ABC$ is similar to $\triangle ACD$. Let P be a point on the extension of DA past A such that $\angle BDC = \angle ACP$. Compute DP^2.

2020 CCA Math Bonanza Lightning Round#4.3

L4.4 A sequence $\{a_n\}$ is defined such that $a_i = i$ for $i = 1, 2, 3 \ldots, 2020$ and for $i > 2020$, a_i is the average of the previous 2020 terms. What is the largest integer less than or equal to $\lim_{n\to\infty} a_n$?

2020 CCA Math Bonanza Lightning Round#4.4

L5.1 Professor Shian Bray is buying CCA Math Bananas™. He starts with $\$500$. The first CCA Math Bananas™ he buys costs $\$1$. Each time after he buys a CCA Math Banana™, the cost of a CCA Math Bananas™ doubles with probability $\frac{1}{2}$ (otherwise staying the same). Professor Bray buys CCA Math Bananas™ until he cannot afford any more, ending with n CCA Math Bananas™. Estimate the expected value of n. An estimate of E earns $2^{1-0.25|E-A|}$ points, where A is the actual answer.

2020 CCA Math Bonanza Lightning Round#5.1

L5.2 A teacher writes the positive integers from 1 to 12 on a blackboard. Every minute, they choose a number k uniformly at random from the written numbers, subtract k from each number $n \geq k$ on the blackboard (without touching the numbers $n < k$), and erase every 0 on the board. Estimate the expected number of minutes that pass before the board is empty. An estimate of E earns $2^{1-0.5|E-A|}$ points, where A is the actual answer.

2020 CCA Math Bonanza Lightning Round#5.2

L5.3 Estimate the number of pairs of integers $1 \leq a, b \leq 1000$ satisfying $\gcd(a, b) = \gcd(a+1, b+1)$. An estimate of E earns $2^{1-0.00002|E-A|}$ points, where A is the actual answer.

2020 CCA Math Bonanza Lightning Round#5.3

L5.4 Submit a positive integer less than or equal to 15. Your goal is to submit a number that is close to the number of teams submitting it. If you submit N and the total number of teams at the competition (including your own team) who submit N is T, your score will be $\frac{2}{0.5|N-T|+1}$.

2020 CCA Math Bonanza Lightning Round#5.4

– Tiebreaker Round

TB1 In a group of 2020 people, some pairs of people are friends (friendship is mutual). It is known that no two people (not necessarily friends) share a friend. What is the maximum number of unordered pairs of people who are friends?

2020 CCA Math Bonanza Tiebreaker Round#1

TB2 Shayan is playing a game by himself. He picks **relatively prime** integers a and b such that $1 < a < b < 2020$. He wins if every integer $m \geq \frac{ab}{2}$ can be expressed in the form $ax + by$ for nonnegative integers x and y. He hasn't been winning often, so he decides to write down all winning pairs (a, b), from (a_1, b_1) to (a_n, b_n). What is $b_1 + b_2 + \ldots + b_n$?

2020 CCA Math Bonanza Tiebreaker Round#2

TB3 How many unordered triples A, B, C of distinct lattice points in $0 \leq x, y \leq 4$ have the property that $2[ABC]$ is an integer divisible by 5?

2020 CCA Math Bonanza Tiebreaker Round#3

TB3 Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. The incircle of ABC meets BC at D. Line AD meets the circle through B, D, and the reflection of C over AD at a point $P \neq D$. Compute AP.

2020 CCA Math Bonanza Tiebreaker Round#4