Finals 1987

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–	Day 1
1	There are $n \geq 2$ points in a square side 1. Show that one can label the points $P_1, P_2, ..., P_n$ such that $\sum_{i=1}^n \|P_{i-1} - P_i\|^2 \leq 4$, where we use cyclic subscripts, so that P_0 means P_n.
2	A regular n-gon is inscribed in a circle radius 1. Let X be the set of all arcs PQ, where P, Q are distinct vertices of the n-gon. 5 elements $L_1, L_2, ..., L_5$ of X are chosen at random (so two or more of the L_i can be the same). Show that the expected length of $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ is independent of n.
3	$w(x)$ is a polynomial with integer coefficients. Let p_n be the sum of the digits of the number $w(n)$. Show that some value must occur infinitely often in the sequence $p_1, p_2, p_3, ...$.
–	Day 2
4	Let S be the set of all tetrahedra which satisfy: (1) the base has area 1, (2) the total face area is 4, and (3) the angles between the base and the other three faces are all equal. Find the element of S which has the largest volume.
5	Find the smallest n such that $n^2 - n + 11$ is the product of four primes (not necessarily distinct).
6	A plane is tiled with regular hexagons of side 1. A is a fixed hexagon vertex. Find the number of paths P such that: (1) one endpoint of P is A, (2) the other endpoint of P is a hexagon vertex, (3) P lies along hexagon edges, (4) P has length 60, and (5) there is no shorter path along hexagon edges from A to the other endpoint of P.