Czech-Polish-Slovak Junior Match 2019

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by parmenides51

–	Individual
1	Find all pairs of positive integers a, b such that $\sqrt{a+2\sqrt{b}} = \sqrt{a-2\sqrt{b}} + \sqrt{b}$.
2	Let ABC be a triangle with centroid T. Denote by M the midpoint of BC. Let D be a point on the ray opposite to the ray BA such that $AB = BD$. Similarly, let E be a point on the ray opposite to the ray CA such that $AC = CE$. The segments TD and TE intersect the side BC in P and Q, respectively. Show that the points P, Q and M split the segment BC into four parts of equal length.
3	Determine all positive integers n such that it is possible to fill the $n \times n$ table with numbers $1, 2$ and -3 so that the sum of the numbers in each row and each column is 0.
4	Let k be a circle with diameter AB. A point C is chosen inside the segment AB and a point D is chosen on k such that BCD is an acute-angled triangle, with circumcentre denoted by O. Let E be the intersection of the circle k and the line BO (different from B). Show that the triangles BCD and ECA are similar.
5	Given is a group in which everyone has exactly d friends and every two strangers have exactly one common friend. Prove that there are at most $d^2 + 1$ people in this group.
–	Team
1	Rational numbers a, b are such that $a + b$ and $a^2 + b^2$ are integers. Prove that a, b are integers.
2	The chess piece *sick rook* can move along rows and columns as a regular rook, but at most by 2 fields. We can place *sick rooks* on a square board in such a way that no two of them attack each other and no field is attacked by more than one *sick rook*. a) Prove that on 30×30 board, we cannot place more than 100 *sick rooks*. b) Find the maximum number of *sick rooks* which can be placed on 8×8 board. c) Prove that on 32×32 board, we cannot place more than 120 *sick rooks*.
3	Let $ABCD$ be a convex quadrilateral with perpendicular diagonals, such that $\angle BAC = \angle ADB$, $\angle CBD = \angle DCA$, $AB = 15$, $CD = 8$. Show that $ABCD$ is cyclic and find the distance between its circumcenter and the intersection point of its diagonals.

4 Determine all possible values of the expression $xy + yz + zx$ with real numbers x, y, z satisfying the conditions $x^2 - yz = y^2 - zx = z^2 - xy = 2$.

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5 Let $A_1A_2...A_{360}$ be a regular 360-gon with centre S. For each of the triangles $A_1A_{50}A_{68}$ and $A_1A_{50}A_{69}$ determine, whether its images under some 120 rotations with centre S can have (as triangles) all the 360 points $A_1, A_2, ..., A_{360}$ as vertices.

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6 Given is a cyclic quadrilateral $ABCD$. Points K, L, M, N lying on sides AB, BC, CD, DA, respectively, satisfy $\angle ADK = \angle BCK$, $\angle BAL = \angle CDL$, $\angle CBM = \angle DAM$, $\angle DCN = \angle ABN$. Prove that lines KM and LN are perpendicular.
