Finals 1985

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–	Day 1
1	Find the largest k such that for every positive integer n we can find at least k numbers in the set $\{n+1, n+2, ..., n+16\}$ which are coprime with $n(n+17)$.
2	Given a square side 1 and $2n$ positive reals $a_1, b_1, ..., a_n, b_n$ each ≤ 1 and satisfying $\sum a_ib_i \geq 100$. Show that the square can be covered with rectangles R_i with sides length (a_i, b_i) parallel to the square sides.
3	The function $f : R \to R$ satisfies $f(3x) = 3f(x) - 4f(x)^3$ for all real x and is continuous at $x = 0$. Show that $\lvert f(x)\rvert \leq 1$ for all x.
–	Day 2
4	P is a point inside the triangle ABC is a triangle. The distance of P from the lines BC, CA, AB is d_a, d_b, d_c respectively. If r is the inradius, show that $$\frac{2}{\frac{1}{d_a}+\frac{1}{d_b}+\frac{1}{d_c}} < r < \frac{d_a+d_b+d_c}{2}$$
5	$p(x, y)$ is a polynomial such that $p(cost, sint) = 0$ for all real t. Show that there is a polynomial $q(x, y)$ such that $p(x, y) = (x^2 + y^2 - 1)q(x, y)$.
6	There is a convex polyhedron with k faces. Show that if more than $k/2$ of the faces are such that no two have a common edge, then the polyhedron cannot have an inscribed sphere.