US Math Competition Association

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by parmenides51, USMCADirector

– Online Qualifier

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1 Find the sum of all positive integers n such that $\frac{2020}{n^3+n}$ is an integer.

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2 Elmo bakes cookies at a rate of one per 5 minutes. Big Bird bakes cookies at a rate of one per 6 minutes. Cookie Monster *consumes* cookies at a rate of one per 4 minutes. Together Elmo, Big Bird, Cookie Monster, and Oscar the Grouch produce cookies at a net rate of one per 8 minutes. How many minutes does it take Oscar the Grouch to bake one cookie?

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3 For a word w consisting of n lowercase letters, an *edit* is specified by a pair (i, c) where $i \in \{1, \ldots, n\}$ and c is a lowercase letter, and transforms w by replacing its i^{th} letter with c. It is possible that c is the same as the letter it replaced.

How many sequences of six edits transform `frog` into `goat`? Note that on the word `abcd`, the edits $(1, \texttt{a})$ and $(2, \texttt{b})$ are considered distinct, even though both result in the word `abcd`.

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4 Let $f(n)$ denote the largest odd factor of n, including possibly n. Determine the value of

$$\frac{f(1)}{1} + \frac{f(2)}{2} + \frac{f(3)}{3} + \cdots + \frac{f(2048)}{2048},$$

rounded to the nearest integer.

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5 A unit square $ABCD$ is balanced on a flat table with only its vertex A touching the table, such that AC is perpendicular to the table. The square loses balance and falls to one side. At the end of the fall, A is in the same place as before, and B is also touching the table. Compute the area swept by the square during its fall.

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6 In the land of Brobdingnag, a parking lot has 2020 parking spaces in a row. 674 cars, each two parking spaces wide, arrive at the parking lot one by one. Each car parks in a pair of consecutive vacant spaces, selected uniformly at random over all such pairs; for example, the first car can park in 2019 ways, all with equal probability. If no pair of consecutive vacant spaces remain when a car arrives, it leaves disappointedly. What is the probability that all 674 cars successfully park?

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7 Compute the value of

$$\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 3\cos\frac{6\pi}{7} + 4\cos\frac{8\pi}{7} + 5\cos\frac{10\pi}{7} + 6\cos\frac{12\pi}{7}.$$

8 Two right cones each have base radius 4 and height 3, such that the apex of each cone is the center of the base of the other cone. Find the surface area of the union of the cones.

9 Find a 7-digit integer divisible by 128, whose decimal representation contains only the digits 2 and 3.

10 Let $ABCD$ be a unit square, and let E be a point on segment AC such that $AE = 1$. Let DE meet AB at F and BE meet AD at G. Find the area of CFG.

11 What is the largest real x satisfying $(x+1)(x+2)(x+3)(x+6) = 2x+1$?

12 Kelvin the Frog is playing the game of Survival. He starts with two fair coins. Every minute, he flips all his coins one by one, and throws a coin away if it shows tails. The game ends when he has no coins left, and Kelvin's score is the *square* of the number of minutes elapsed. What is the expected value of Kelvin's score? For example, if Kelvin flips two tails in the first minute, the game ends and his score is 1.

13 Ω is a quarter-circle of radius 1. Let O be the center of Ω, and A and B be the endpoints of its arc. Circle ω is inscribed in Ω. Circle γ is externally tangent to ω and internally tangent to Ω on segment OA and arc AB. Determine the radius of γ.

14 Call a real number *amiable* if it can be expressed in the form $a - b\sqrt{2}$, where $1 \le a, b \le 100$ are integers. Find the amiable number x that minimizes $\left|x - \frac{1}{3}\right|$.

15 The number $2020!$ can be expressed as $7^k \cdot m$, where k, m are integers and m is not divisible by 7. Find the remainder when m is divided by 49.

16 How many paths from $(0,0)$ to $(2020, 2020)$, consisting of unit steps up and to the right, pass through at most one point with both coordinates even, other than $(0,0)$ and $(2020, 2020)$?

17 Let $P(x)$ be the product of all linear polynomials $ax + b$, where $a, b \in \{0, \dots, 2016\}$ and $(a, b) \neq (0, 0)$. Let $R(x)$ be the remainder when $P(x)$ is divided by $x^5 - 1$. Determine the remainder when $R(5)$ is divided by 2017.

18 Kelvin the Frog writes 2020 words on a blackboard, with each word chosen uniformly randomly from the set $\{\texttt{happy}, \texttt{boom}, \texttt{swamp}\}$. A multiset of seven words is *merry* if its elements can spell "`happy happy boom boom swamp swamp swamp`." For example, the eight words

$$\texttt{swamp}, \texttt{happy}, \texttt{boom}, \texttt{swamp}, \texttt{swamp}, \texttt{boom}, \texttt{swamp}, \texttt{happy}$$

contain four merry multisets. Determine the expected number of merry multisets contained in the words on the blackboard.

http://www.hpmor.com/chapter/12

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19 Let x_1, x_2, x_3 be the solutions to $(x-13)(x-33)(x-37) = 1337$. Find the value of

$$\sum_{i=1}^{3} \left[(x_i-13)^3 + (x_i-33)^3 + (x_i-37)^3\right].$$

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20 Let Ω be a circle centered at O. Let $ABCD$ be a quadrilateral inscribed in Ω, such that $AB = 12$, $AD = 18$, and AC is perpendicular to BD. The circumcircle of AOC intersects ray DB past B at P. Given that $\angle PAD = 90^\circ$, find BD^2.

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21 The sequence $a_1, a_2, \ldots$ is defined by $a_1 = 2019$, $a_2 = 2020$, $a_3 = 2021$, $a_{n+3} = a_n(a_{n+1}a_{n+2} + 1)$ for $n \geq 1$. Determine the value of the infinite sum

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

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22 Kelvin the Frog places 40 rooks on a uniformly random subset of 40 squares of a 20×20 chessboard. Then, Alex the Kat chooses two of the 40 rooks uniformly at random. What is the probability that Alex's two rooks attack each other? Two rooks attack each other if they are on the same row or column, and no piece stands between them.

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23 The sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are defined by $a_1 = \frac{5}{2}\sqrt[3]{2}$, $b_1 = 2\sqrt[3]{4}$, and for $n \geq 1$, $a_{n+1} = a_n^2 - 2b_n$, $b_{n+1} = b_n^2 - 2a_n$. There exist real numbers u, v such that

$$\lim_{n\to\infty} \frac{a_n}{ub_n^v} = 1.$$

Determine the pair (u, v).

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24 Compute the value of

$$\sum_{i=0}^{2026} \frac{i^2}{9+i^4} \pmod{2027},$$

where $\frac{1}{a}$ denotes the multiplicative inverse of a modulo 2027.

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25 Let AB be a segment of length 2. The locus of points P such that the P-median of triangle ABP and its reflection over the P-angle bisector of triangle ABP are perpendicular determines some region R. Find the area of R.

– National Championship Challenger

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1 If U, S, M, C, A are distinct (not necessarily positive) integers such that $U \cdot S \cdot M \cdot C \cdot A = 2020$, what is the greatest possible value of $U + S + M + C + A$?

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2 Sarah is fighting a dragon in DnD. She rolls two fair twenty-sided dice numbered $1, 2, \dots, 20$. She vanquishes the dragon if the product of her two rolls is a multiple of 4. What is the probability that the dragon is vanquished?

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3 If $x(y+1) = 41$ and $x^2(y^2+1) = 881$, determine all possible pairs of real numbers (x, y).

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4 Let $ABCDEF$ be a regular hexagon with side length two. Extend FE and BD to meet at G. Compute the area of $ABGF$.

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5 Call a positive integer n an $A - B$ number if the base A and base B representations of n are three-digit numbers that are reverses of each other. For example, 87 is a $5 - 6$ number because $87 = 223_6 = 322_5$. Compute the sum of all $7 - 11$ numbers.

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6 Alex is thinking of a number that is divisible by all of the positive integers 1 through 200 inclusive except for two consecutive numbers. What is the smaller of these numbers?

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7 Jenn is competing in a puzzle hunt with six regular puzzles and one additional meta-puzzle. Jenn can solve any puzzle regularly. Additionally, if she has already solved the meta-puzzle, Jenn can also back-solve a puzzle. A back-solve is distinguishable from a regular solve. The meta puzzle cannot be the first puzzle solved. How many possible solve orders for the seven puzzles are possible?

For example, Jenn may solve#3, solve#5, solve#6, solve the meta-puzzle, solve#2, solve#1, and then solve#4.
However, she may not solve#2, solve#4, solve#6, back-solve#1, solve#3, solve#5, and then solve the meta-puzzle.

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8 Two altitudes of a triangle have lengths 8 and 15. How many possible integer lengths are there for the third altitude?

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9 Let Ω be a unit circle and A be a point on Ω. An angle $0 < \theta < 180^\circ$ is chosen uniformly at random, and Ω is rotated θ degrees clockwise about A. What is the expected area swept by this rotation?

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10 If $0 < x < \frac{\pi}{2}$ and $\frac{\sin x}{1+\cos x} = \frac{1}{3}$, what is $\frac{\sin 2x}{1+\cos 2x}$?

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11 A permutation of $USMCAUSMCA$ is selected uniformly at random. What is the probability that this permutation is exactly one transposition away from $USMCAUSMCA$ (i.e. does not equal $USMCAUSMCA$, but can be turned into $USMCAUSMCA$ by swapping one pair of letters)?

12 Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^2}{d^3+2}.$$

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13 Equiangular octagon $ABCDEFGH$ is inscribed in a circle centered at O. Chords AD and BG intersect at K. Given that $AB = 2$ and the octagon has area 15, compute the area of $HAKBO$.

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14 Kelvin the Nanofrog is visiting his friend, Alex the Nanokat, who lives 483 nanometers away. On his trip to Alex's home, Kelvin travels at k nanometers an hour, where k is an integer, and completes the trip in an integer number of minutes. On his return journey, he travels slower by 7 nanometers an hour, and completes the trip in an integer number of minutes. What is the smallest total number of minutes Kelvin could have spent traveling?

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15 Find the greatest prime factor of $2^{56} + (2^{15}+1)(2^{29}+2^{15}+1)$.

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16 Triangle ABC has $BC = 7, CA = 8, AB = 9$. Let D, E, F be the midpoints of BC, CA, AB respectively, and let G be the intersection of AD and BE. G' is the reflection of G across D. Let $G'E$ meet CG at P, and let $G'F$ meet BG at Q. Determine the area of $APG'Q$.

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17 An *island* is a contiguous set of at least two equal digits. Let $b(n)$ be the number of islands in the binary representation of n. For example, $2020_{10} = 11111100100_2$, so $b(2020) = 3$. Compute

$$b(1) + b(2) + \cdots + b(2^{2020}).$$

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18 Alice, Bob, Chad, and Denise decide to meet for a virtual group project between 1 and 3 PM, but they don't decide what time. Each of the four group members sign on to Zoom at a uniformly random time between 1 and 2 PM, and they stay for 1 hour. The group gets work done whenever at least three members are present. What is the expected number of minutes that the group gets work done?

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19 Call a right triangle *peri-prime* if it has relatively prime integer side lengths, perimeter a multiple of 65, and at least one leg with length less than 100. Compute the sum of all possible lengths for the smallest leg of a peri-prime triangle.

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20 Yu Semo and Yu Sejmo have created sequences of symbols $\mathcal{U} = (\mathrm{U}_1, \ldots, \mathrm{U}_6)$ and $\mathcal{J} = (\mathrm{J}_1, \ldots, \mathrm{J}_6)$. These sequences satisfy the following properties.

- Each of the twelve symbols must be Σ, $\#$, $\triangle$, or $\mathbb{Z}$.
- In each of the sets $\{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_4, \mathrm{U}_5\}$, $\{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_4, \mathrm{J}_5\}$, $\{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3\}$, $\{\mathrm{U}_4, \mathrm{U}_5, \mathrm{U}_6\}$, $\{\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3\}$, $\{\mathrm{J}_4, \mathrm{J}_5, \mathrm{J}_6\}$,

no two symbols may be the same.
- If integers $d \in \{0,1\}$ and $i,j \in \{1,2,3\}$ satisfy $\text{U}_{i+3d} = \text{J}_{j+3d}$, then $i < j$.

How many possible values are there for the pair $(\mathcal{U}, \mathcal{J})$?

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21 Let $ABCDEF$ be a regular octahedron with unit side length, such that $ABCD$ is a square. Points G, H are on segments BE, DF respectively. The planes AGD and BCH divide the octahedron into three pieces, each with equal volume. Compute BG.

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22 Carol places a king on a 5×5 chessboard. The king starts on the lower-left corner, and each move it steps one square to the right, up, up-right, up-left, or down-right. How many ways are there for the king to get to the top-right corner without visiting the same square twice?

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23 Let f_n be a sequence defined by $f_0 = 2020$ and

$$f_{n+1} = \frac{f_n + 2020}{2020 f_n + 1}$$

for all $n \geq 0$. Determine f_{2020}.

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24 Farmer John has a 47×53 rectangular square grid. He labels the first row $1, 2, \cdots, 47$, the second row $48, 49, \cdots, 94$, and so on. He plants corn on any square of the form $47x + 53y$, for non-negative integers x, y. Given that the unplanted squares form a contiguous region R, find the perimeter of R.

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25 Let $S = \{1, \cdots, 6\}$ and $\mathcal{P}$ be the set of all nonempty subsets of S. Let N equal the number of functions $f : \mathcal{P} \to S$ such that if $A, B \in \mathcal{P}$ are disjoint, then $f(A) \neq f(B)$.
Determine the number of positive integer divisors of N.

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26 Let Γ be a circle centered at O with chord AB. The tangents to Γ at A and B meet at C. A secant from C intersects chord AB at D and Γ at E such that D lies on segment CE. Given that $\angle BOD + \angle EAD = 180^\circ$, $AE = 1$, and $BE = 2$, find CE.

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27 Let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Evaluate

$$\lim_{m \to \infty} \frac{\sum_{n=1}^{m} \phi(60n)}{\sum_{n=1}^{m} \phi(n)}$$

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28 Call a polynomial f with positive integer coefficients *triangle-compatible* if any three coefficients of f satisfy the triangle inequality. For instance, $3x^3 + 4x^2 + 6x + 5$ is triangle-compatible, but $3x^3 + 3x^2 + 6x + 5$ is not. Given that f is a degree 20 triangle-compatible polynomial with -20 as a root, what is the least possible value of $f(1)$?

Note: this problem is also Premier#3

29 Let ABC be a triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Given that $AB = 8, AC = 10$, and $\angle BAC = 60^\circ$, find the area of $BCHG$.

Note: this is a modified version of Premier#2

30 For a positive integer n, let $\Omega(n)$ denote the number of prime factors of n, counting multiplicity. Let $f_1(n)$ and $f_3(n)$ denote the sum of positive divisors $d|n$ where $\Omega(d) \equiv 1 \pmod 4$ and $\Omega(d) \equiv 3 \pmod 4$, respectively. For example, $f_1(72) = 72 + 2 + 3 = 77$ and $f_3(72) = 8 + 12 + 18 = 38$. Determine $f_3(6^{2020}) - f_1(6^{2020})$.

– National Championship Premier

1 Let $\mathcal{P}$ be a finite set of squares on an infinite chessboard. Kelvin the Frog notes that $\mathcal{P}$ may be tiled with only 1×2 dominoes, while Alex the Kat notes that $\mathcal{P}$ may be tiled with only 2×1 dominoes. The dominoes cannot be rotated in each tiling. Prove that the area of $\mathcal{P}$ is a multiple of 4.

2 Let ABC be an acute triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Show that $BCHG$ is a parallelogram.

3 Call a polynomial f with positive integer coefficients *triangle-compatible* if any three coefficients of f satisfy the triangle inequality. For instance, $3x^3 + 4x^2 + 6x + 5$ is triangle-compatible, but $3x^3 + 3x^2 + 6x + 5$ is not. Given that f is a degree 20 triangle-compatible polynomial with -20 as a root, what is the least possible value of $f(1)$?

4 Suppose $n > 1$ is an odd integer satisfying $n \mid 2^{\frac{n-1}{2}} + 1$. Prove or disprove that n is prime.

Note: unfortunately, the original form of this problem did not include the red text, rendering it unsolvable. We sincerely apologize for this error and are taking concrete steps to prevent similar issues from reoccurring, including computer-verifying problems where possible. All teams will receive full credit for the question.

5 Alex the Kat and Kelvin the Frog play a game on a complete graph with n vertices. Kelvin goes first, and the players take turns selecting either a single edge to remove from the graph, or a single vertex to remove from the graph. Removing a vertex also removes all edges incident to that vertex. The player who removes the final vertex wins the game. Assuming both players play perfectly, for which positive integers n does Kelvin have a winning strategy?

6 Let P be a non-constant polynomial with integer coefficients such that if n is a perfect power, so is $P(n)$. Prove that $P(x) = x$ or P is a perfect power of a polynomial with integer coefficients.

A perfect power is an integer n^k, where $n \in \mathbb{Z}$ and $k \geq 2$. A perfect power of a polynomial is a polynomial $P(x)^k$, where P has integer coefficients and $k \geq 2$.

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7 Let $ABCD$ be a convex quadrilateral, and let ω_A and ω_B be the incircles of $\triangle ACD$ and $\triangle BCD$, with centers I and J. The second common external tangent to ω_A and ω_B touches ω_A at K and ω_B at L. Prove that lines AK, BL, IJ are concurrent.

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8 Let n, m be positive integers, and let α be an irrational number satisfying $1 < \alpha < n$. Define the set

$$X = \{a + b\alpha : 0 \leq a \leq n \text{ and } 0 \leq b \leq m\}.$$

Let $x_0 \leq x_1 \leq \cdots \leq x_{(n+1)(m+1)-1}$ be the elements of X. Show that for all $i + j \leq (n+1)(m+1) - 1$, we have that $x_{i+j} \leq x_i + x_j$.
