ITAMO 2000

www.artofproblemsolving.com/community/c1056673

by parmenides51

1 A possitive integer is called *special* if all its decimal digits are equal and it can be represented as the sum of squares of three consecutive odd integers.
(a) Find all 4-digit *special* numbers
(b) Are there 2000-digit *special* numbers?

---

2 Let $ABCD$ be a convex quadrilateral, and write $\alpha = \angle DAB$, $\beta = \angle ADB$, $\gamma = \angle ACB$, $\delta = \angle DBC$ and $\epsilon = \angle DBA$. Assuming that $\alpha < \pi/2$, $\beta + \gamma = \pi/2$, and $\delta + 2\epsilon = \pi$, prove that $(DB + BC)^2 = AD^2 + AC^2$.

---

3 A pyramid with the base $ABCD$ and the top V is inscribed in a sphere. Let $AD = 2BC$ and let the rays AB and DC intersect in point E. Compute the ratio of the volume of the pyramid $VAED$ to the volume of the pyramid $VABCD$.

---

4 Let $n > 1$ be a fixed integer. Alberto and Barbara play the following game:
(i) Alberto chooses a positive integer,
(ii) Barbara chooses an integer greater than 1 which is a multiple or submultiple of the number Alberto chose (including itself),
(iii) Alberto increases or decreases the Barbara's number by 1.
Steps (ii) and (iii) are alternatively repeated. Barbara wins if she succeeds to reach the number n in at most 50 moves. For which values of n can she win, no matter how Alberto plays?

---

5 A man disposes of sufficiently many metal bars of length 2 and wants to construct a grill of the shape of an $n \times n$ unit net. He is allowed to fold up two bars at an endpoint or to cut a bar into two equal pieces, but two bars may not overlap or intersect. What is the minimum number of pieces he must use?

---

6 Let $p(x)$ be a polynomial with integer coefficients such that $p(0) = 0$ and $0 \leq p(1) \leq 10^7$. Suppose that there exist positive integers a, b such that $p(a) = 1999$ and $p(b) = 2001$. Determine all possible values of $p(1)$.

(Note: 1999 is a prime number.)
