Dutch Mathematical Olympiad 1990

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by moldovan

1 Prove that for every integer $n > 1, 1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1) < n^n$.

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2 Consider the sequence $a_1 = \frac{3}{2}, a_{n+1} = \frac{3a_n^2+4a_n-3}{4a_n^2}$. (a) Prove that $1 < a_n$ and $a_{n+1} < a_n$ for all n. (b) From (a) it follows that $\lim\limits_{n\to\infty} a_n$ exists. Find this limit. (c) Determine $\lim\limits_{n\to\infty} a_1a_2a_3...a_n$.

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3 A polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx$ with $a, b, c, d > 0$ is such that $f(x)$ is an integer for $x \in \{-2, -1, 0, 1, 2\}$ and $f(1) = 1$ and $f(5) = 70$. (a) Show that $a = \frac{1}{24}, b = \frac{1}{4}, c = \frac{11}{24}, d = \frac{1}{4}$. (b) Prove that $f(x)$ is an integer for all $x \in \mathbb{Z}$.

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4 If $ABCDEFG$ is a regular 7-gon with side 1, show that: $\frac{1}{AC} + \frac{1}{AD} = 1$.
