Dutch Mathematical Olympiad 1991

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1 Prove that for any three positive real numbers a, b, c, $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2} \cdot \frac{1}{a+b+c}$.

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2 An angle with vertex A and measure α and a point P_0 on one of its rays are given so that $AP_0 = 2$. Point P_1 is chose on the other ray. The sequence of points $P_1, P_2, P_3, ...$ is defined so that P_n lies on the segment AP_{n-2} and the triangle $P_nP_{n-1}P_{n-2}$ is isosceles with $P_nP_{n-1} = P_nP_{n-2}$ for all $n \geq 2$. (a) Prove that for each value of α there is a unique point P_1 for which the sequence $P_1, P_2, ..., P_n, ...$ does not terminate. (b) Suppose that the sequence $P_1, P_2, ...$ does not terminate and that the length of the polygonal line $P_0P_1P_2...P_k$ tends to 5 when $k \to \infty$. Compute the length of P_0P_1.

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3 A real function f satisfies $4f(f(x)) - 2f(x) - 3x = 0$ for all real numbers x. Prove that $f(0) = 0$.

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4 Three real numbers a, b, c satisfy the equations $a + b + c = 3, a^2 + b^2 + c^2 = 9, a^3 + b^3 + c^3 = 24$. Find $a^4 + b^4 + c^4$.

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5 Let H be the orthocenter, O the circumcenter, and R the circumradius of an acute-angled triangle ABC. Consider the circles k_a, k_b, k_c, k_h, k, all with radius R, centered at A, B, C, H, M, respectively. Circles k_a and k_b meet at M and F; k_a and k_c meet at M and E; and k_b and k_c meet at M and D. (a) Prove that the points D, E, F lie on the circle k_h. (b) Prove that the set of the points inside k_h that are inside exactly one of the circles k_a, k_b, k_c has the area twice the area of $\triangle ABC$.
