

**Federal Competition For Advanced Students, Part 2 1997**
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## – Day 1

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- 1** Let  $a$  be a fixed integer. Find all integer solutions  $x, y, z$  of the system:  
 $5x + (a+2)y + (a+2)z = a$ ,  $(2a+4)x + (a^2+3)y + (2a+2)z = 3a-1$ ,  $(2a+4)x + (2a+2)y + (a^2+3)z = a+1$ .
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- 2** A positive integer  $K$  is given. Define the sequence  $(a_n)$  by  $a_1 = 1$  and  $a_n$  is the  $n$ -th natural number greater than  $a_{n-1}$  which is congruent to  $n$  modulo  $K$ . (a) Find an explicit formula for  $a_n$ . (b) What is the result if  $K = 2$ ?
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- 3** Let be given a triangle  $ABC$ . Points  $P$  on side  $AC$  and  $Y$  on the production of  $CB$  beyond  $B$  are chosen so that  $Y$  subtends equal angles with  $AP$  and  $PC$ . Similarly,  $Q$  on side  $BC$  and  $X$  on the production of  $AC$  beyond  $C$  are such that  $X$  subtends equal angles with  $BQ$  and  $QC$ . Lines  $YP$  and  $XB$  meet at  $R$ ,  $XQ$  and  $YA$  meet at  $S$ , and  $XB$  and  $YA$  meet at  $D$ . Prove that  $PQRS$  is a parallelogram if and only if  $ACBD$  is a cyclic quadrilateral.
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## – Day 2

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- 4** Determine all quadruples  $(a, b, c, d)$  of real numbers satisfying the equation:  
 $256a^3b^3c^3d^3 = (a^6 + b^2 + c^2 + d^2)(a^2 + b^6 + c^2 + d^2)(a^2 + b^2 + c^6 + d^2)(a^2 + b^2 + c^2 + d^6)$ .
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- 5** We define the following operation which will be applied to a row of bars being situated side-by-side on positions  $1, 2, \dots, N$ . Each bar situated at an odd numbered position is left as is, while each bar at an even numbered position is replaced by two bars. After that, all bars will be put side-by-side in such a way that all bars form a new row and are situated on positions  $1, \dots, M$ . From an initial number  $a_0 > 0$  of bars there originates a sequence  $(a_n)_{n \geq 0}$ , where  $a_n$  is the number of bars after having applied the operation  $n$  times. (a) Prove that for no  $n > 0$  can we have  $a_n = 1997$ . (b) Determine all natural numbers that can only occur as  $a_0$  or  $a_1$ .
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- 6** For every natural number  $n$ , find all polynomials  $x^2 + ax + b$ , where  $a^2 \geq 4b$ , that divide  $x^{2n} + ax^n + b$ .
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