Federal Competition For Advanced Students, Part 2 1991

www.artofproblemsolving.com/community/c1060497

by parmenides51, moldovan

– Day 1

1 Consider a convex solid K in space and two parallel planes ϵ_1 and ϵ_2 on the distance 1 tangent to K. A plane ϵ between ϵ_1 and ϵ_2 is on the distance d_1 from ϵ_1. Find all d_1 such that the part of K between ϵ_1 and ϵ always has a volume not exceeding half the volume of K.

2 Find all functions $f : \mathbb{Z} - \{0\} \to \mathbb{Q}$ satisfying:
$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{2}$, whenever $x, y, \frac{x+y}{3} \in \mathbb{Z} - \{0\}$.

3 (a) Prove that 91 divides $n^{37} - n$ for all integers n. (b) Find the largest k that divides $n^{37} - n$ for all integers n.

– Day 2

4 The sequence (a_n) is given by $a_1 = 1, a_2 = 0$ and:
$a_{2k+1} = a_k + a_{k+1}, a_{2k+2} = 2a_{k+1}$ for $k \in \mathbb{N}$.

Find a_m for $m = 2^{19} + 91$.

5 For all positive integers n prove the inequality:
$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n$.

6 Find the number of ten-digit natural numbers (which do not start with zero) containing no block 1991.