Federal Competition For Advanced Students, Part 2 1988

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– Day 1

1 If $a_1, ..., a_{1988}$ are positive numbers whose arithmetic mean is 1988, show that:
$$\sqrt[1988]{\prod_{i,j=1}^{1988}\left(1+\frac{a_i}{a_j}\right)} \ge 2^{1988}$$
and determine when equality holds.

2 An equilateral triangle $A_1A_2A_3$ is divided into four smaller equilateral triangles by joining the midpoints A_4, A_5, A_6 of its sides. Let $A_7, ..., A_{15}$ be the midpoints of the sides of these smaller triangles. The 15 points $A_1, ..., A_{15}$ are colored either green or blue. Show that with any such colouring there are always three mutually equidistant points A_i, A_j, A_k having the same color.

3 Show that there is precisely one sequence $a_1, a_2, ...$ of integers which satisfies $a_1 = 1, a_2 > 1$, and $a_{n+1}^3 + 1 = a_n a_{n+2}$ for $n \ge 1$.

– Day 2

4 Let a_{ij} be nonnegative integers such that $a_{ij} = 0$ if and only if $i > j$ and that $\sum_{j=1}^{1988} a_{ij} = 1988$ holds for all $i = 1, ..., 1988$. Find all real solutions of the system of equations:
$$\sum_{j=1}^{1988}(1 + a_{ij})x_j = i + 1, 1 \le i \le 1988.$$

5 The bisectors of angles B and C of triangle ABC intersect the opposite sides in points B' and C' respectively. Show that the line $B'C'$ intersects the incircle of the triangle.

6 Determine all monic polynomials $p(x)$ of fifth degree having real coefficients and the following property: Whenever a is a (real or complex) root of $p(x)$, then so are $\frac{1}{a}$ and $1 - a$.