Federal Competition For Advanced Students, Part 2 1987

www.artofproblemsolving.com/community/c1060509

by parmenides51, moldovan

–	Day 1
1	The sides a, b and the bisector of the included angle γ of a triangle are given. Determine necessary and sufficient conditions for such triangles to be constructible and show how to reconstruct the triangle.
2	Find the number of all sequences $(x_1, ..., x_n)$ of letters a, b, c that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $1 \leq i \leq n-1$.
3	Let $x_1, ..., x_n$ be positive real numbers. Prove that: $\sum_{k=1}^{n} x_k + \sqrt{\sum_{k=1}^{n} x_k^2} \leq \frac{n+\sqrt{n}}{n^2} \left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{k=1}^{n} x_k^2\right).$
–	Day 2
4	Find all triples (x, y, z) of natural numbers satisfying $2xz = y^2$ and $x + z = 1987$.
5	Let P be a point in the interior of a convex n-gon $A_1A_2...A_n$ $(n \geq 3)$. Show that among the angles $\beta_{ij} = \angle A_iPA_j, 1 \leq i \leq n$, there are at least $n-1$ angles satisfying $90^\circ \leq \beta_{ij} \leq 180^\circ$.
6	Determine all polynomials $P_n(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ with integer coefficients whose n zeros are precisely the numbers $a_1, ..., a_n$ (counted with their respective multiplicities).