Olympic Revenge 2020

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by ZeusDM

1 Let n be a positive integer and $a_1, a_2, \dots, a_n$ non-zero real numbers. What is the least number of non-zero coefficients that the polynomial $P(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ can have?

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2 For a positive integer n, we say an n-*shuffling* is a bijection $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that there exist exactly two elements i of $\{1, 2, \dots, n\}$ such that $\sigma(i) \neq i$.

Fix some three pairwise distinct n-shufflings $\sigma_1, \sigma_2, \sigma_3$. Let q be any prime, and let $\mathbb{F}_q$ be the integers modulo q. Consider all functions $f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q$ that satisfy, for all integers i with $1 \leq i \leq n$ and all $x_1, \dots x_{i-1}, x_{i+1}, \dots, x_n, y, z \in \mathbb{F}_q^n$,

$$f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) + f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i-1}, y+z, x_{i+1}, \dots, x_n),$$

and that satisfy, for all $x_1, \dots, x_n \in \mathbb{F}_q^n$ and all $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$,

$$f(x_1, \dots, x_n) = -f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

For a given tuple $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$, let $g(x_1, \dots, x_n)$ be the number of different values of $f(x_1, \dots, x_n)$ over all possible functions f satisfying the above conditions.

Pick $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$ uniformly at random, and let $\varepsilon(q, \sigma_1, \sigma_2, \sigma_3)$ be the expected value of $g(x_1, \dots, x_n)$. Finally, let

$$\kappa(\sigma_1, \sigma_2, \sigma_3) = -\lim_{q \to \infty} \log_q \left(-\ln \left(\frac{\varepsilon(q, \sigma_1, \sigma_2, \sigma_3) - 1}{q - 1} \right) \right).$$

Pick three pairwise distinct n-shufflings $\sigma_1, \sigma_2, \sigma_3$ uniformly at random from the set of all n-shufflings. Let $\pi(n)$ denote the expected value of $\kappa(\sigma_1, \sigma_2, \sigma_3)$. Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients such that $q(-3) \neq 0$ and such that $\pi(n) = \frac{p(n)}{q(n)}$ for infinitely many positive integers n. Compute $\frac{p(-3)}{q(-3)}$.

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3 Let ABC be a triangle and ω its circumcircle. Let D and E be the feet of the angle bisectors relative to B and C, respectively. The line DE meets ω at F and G. Prove that the tangents to ω through F and G are tangents to the excircle of $\triangle ABC$ opposite to A.

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4 Let n be a positive integer and A a set of integers such that the set $\{x = a + b \mid a, b \in A\}$ contains $\{1^2, 2^2, \dots, n^2\}$. Prove that there is a positive integer N such that if $n \geq N$, then $|A| > n^{0.666}$.

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5 Let n be a positive integer. Given n points in the plane, prove that it is possible to draw an angle with measure $\frac{2\pi}{n}$ with vertex as each one of the given points, such that any point in the plane is covered by at least one of the angles.