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by parmenides51

1 Show that $3x^5 + 5x^3 - 8x$ is divisible by 120 for any integer x

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2 A tetrahedron has the property that the three segments connecting the pairs of midpoints of opposite edges are equal and mutually orthogonal. Prove that this tetrahedron is regular.

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3 Show how to construct (by a ruler and a compass) a right-angled triangle, given its inradius and circumradius.

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4 Given $I_0 = \{-1, 1\}$, define I_n recurrently as the set of solutions x of the equations $x^2 - 2xy + y^2 - 4^n = 0$,
where y ranges over all elements of I_{n-1}. Determine the union of the sets I_n over all nonnegative integers n.

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5 Let $a_1, a_2, ...$ and $b_1, b_2, ...$ be two arbitrary infinite sequences of natural numbers.
Prove that there exist different indices r and s such that $a_r \geq a_s$ and $b_r \geq b_s$.

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6 There are three balls of distinct colors in a bag. We repeatedly draw out the balls one by one, the balls are put back into the bag after each drawing. What is the probability that, after n drawings,
(a) exactly one color occured?
(b) exactly two oclors occured?
(c) all three colors occured?

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7 A square paper of side n is divided into n^2 unit square cells. A maze is drawn on the paper with unit walls between some cells in such a way that one can reach every cell from every other cell not crossing any wall. Find, in terms of n, the largest possible total length of the walls.
